

SOME RESULTS ON THE LARGE TIME BEHAVIOR OF WEAKLY COUPLED SYSTEMS OF FIRST-ORDER HAMILTON-JACOBI EQUATIONS

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ABSTRACT. Systems of Hamilton-Jacobi equations arise naturally when we study the optimal control problems with pathwise deterministic trajectories with random switching. In this work, we are interested in the large time behavior of weakly coupled systems of first-order Hamilton-Jacobi equations in the periodic setting. The large time behavior for systems of Hamilton-Jacobi equations have been obtained by Camilli-Loreti-Ley and the author (2012) and Mitake-Tran (2012) under quite strict conditions. In this work, we use a PDE approach to extend the convergence result proved by Barles-Souganidis (2000) in the scalar case. This general result permits us to treat lot of general cases, for instance, systems with nonconvex Hamiltonians and systems with strictly convex Hamiltonians. We also obtain some other convergence results under different assumptions, these results give a clearer view on the large time behavior for systems of Hamilton-Jacobi equations.

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1. INTRODUCTION

1.1. Statement of the problem and recalls of the results for single equations. In this paper, we study the large time behavior of systems of Hamilton-Jacobi equations

$$(1.1) \quad \begin{cases} \frac{\partial u_i}{\partial t} + H_i(x, Du_i) + \sum_{j=1}^m d_{ij}(x)u_j = 0 & (x, t) \in \mathbb{T}^N \times (0, +\infty), \\ u_i(x, 0) = u_{0i}(x) & x \in \mathbb{T}^N, \end{cases} \quad i = 1, \dots, m,$$

where \mathbb{T}^N is the N -dimensional torus, the coupling is linear and monotone, i.e.,

$$(1.2) \quad \begin{aligned} & d_{ij} : \mathbb{T}^N \rightarrow \mathbb{R} \text{ are continuous and, for all } x \in \mathbb{T}^N, \\ & d_{ii}(x) \geq 0, \quad d_{ij}(x) \leq 0 \text{ for } i \neq j \quad \text{and} \quad \sum_{j=1}^m d_{ij}(x) = 0 \text{ for all } i \end{aligned}$$

and

$$H_i \in C(\mathbb{T}^N \times \mathbb{R}^N), \quad u_{0i} \in C(\mathbb{T}^N), \quad i = 1, \dots, m.$$

The aim of this work is to improve the first results obtained by Camilli-Ley-Loreti and the author [6] and Mitake-Tran [22] and, more generally, to generalize to systems of the form (1.1), the existing results for the case of a single Hamilton-Jacobi equation

$$(1.3) \quad \begin{cases} \frac{\partial u}{\partial t} + H(x, Du) = 0, & (x, t) \in \mathbb{T}^N \times (0, +\infty), \\ u(x, 0) = u_0(x) & x \in \mathbb{T}^N. \end{cases}$$

Let us start by recalling the existing results for (1.3). The large time behavior has been extensively investigated using both PDE methods and dynamical approaches. The desired result is that to find a unique constant $c \in \mathbb{R}$, the so-called critical value or ergodic constant, and a solution v of the stationary equation

$$(1.4) \quad H(x, Dv) = c \quad \text{in } \mathbb{R}^N$$

such that

$$(1.5) \quad u(x, t) + ct \rightarrow v(x) \text{ uniformly as } t \text{ tends to infinity.}$$

The first results are of Fathi [10] and Namah-Roquejoffre [25] where the convexity of Hamiltonians plays a key role. The result of [10] was proved, in a periodic setting, under the assumption that H is uniformly convex with respect to p , i.e., there exists a constant $\alpha > 0$ such that

$$(1.6) \quad D_{pp}^2 H(x, p) \geq \alpha I, \text{ for all } (x, p) \in \mathbb{T}^N \times \mathbb{R}^N.$$

This result was extended to general strictly convex Hamiltonians in Davini-Siconolfi [8]. We say that a Hamiltonian $H(x, p)$ is strictly convex when, for any $x \in \mathbb{T}^N$,

$$H(x, \lambda p + (1 - \lambda)q) < \lambda H(x, p) + (1 - \lambda)H(x, q) \text{ for any } 0 < \lambda < 1, \quad p \neq q.$$

In [25], the result was proved for Hamiltonians of the form

$$H(x, p) = F(x, p) - f(x), \quad \text{with } F(x, p) \geq F(x, 0) = 0, \quad f \geq 0,$$

$F \in C(\mathbb{T}^N \times \mathbb{R}^N)$, $f \in C(\mathbb{T}^N)$ and F is coercive and convex with respect to p . In this framework, the set

$$\mathcal{F}_{scalar} = \{x_0 \in \mathbb{T}^N : f(x_0) = 0\},$$

which is assumed to be nonempty, plays a crucial role. It appears to be a *uniqueness set* for the stationary equation (1.4), i.e., the solution of (1.4) is uniquely characterized by its value on this set. The idea of [25] is to prove first the convergence of $u(\cdot, t)$ as $t \rightarrow +\infty$ on this set \mathcal{F}_{scalar} and then, to obtain the convergence everywhere by using the limit values of $u(\cdot, t)$ on \mathcal{F}_{scalar} as Dirichlet boundary conditions for (1.4).

Barles and Souganidis [4] succeeded in relaxing a bit the convexity condition on H . Under suitable sets of assumptions, which include the cases of [10, 25], they obtain the convergence (1.5). We state a result of [4] we will be interested in. We assume that (1.4) is solved for $c = 0$ and we introduce the assumptions on H (in general, the assumptions are made on $H - c$)

$$(1.7) \quad \left\{ \begin{array}{l} (i) \text{ The function } p \mapsto H(x, p) \text{ is differentiable a.e. in } x \in \mathbb{T}^N, \\ (ii) \text{ There exists a, possibly empty, compact set } K \text{ of } \mathbb{R}^N \text{ such that:} \\ \quad (a) \ H(x, p) \geq 0 \text{ on } K \times \mathbb{R}^N, \\ \quad (b) \text{ If } H(x, p) \geq \eta > 0 \text{ and } d(x, K) \geq \eta, \text{ then } H_p(x, p)p - H(x, p) \geq \Psi(\eta) > 0. \end{array} \right.$$

If $K = \emptyset$, we define $d(x, K) = +\infty$ for any $x \in \mathbb{T}^N$.

Theorem 1.1. ([4]) *Assume that $H \in C(\mathbb{T}^N \times \mathbb{R}^N)$ and (1.7) holds. Then, any solution $u \in W^{1,\infty}(\mathbb{T}^N \times (0, \infty))$ of (1.3) converges uniformly to a solution $v \in W^{1,\infty}(\mathbb{T}^N)$ of (1.4).*

In this work, we are mainly concerned in extending the previous results to systems. Let us mention Fathi [11], Roquejoffre [26] for other related results in the periodic setting. Some of these results have been also extended beyond the periodic setting, see Barles and Roquejoffre [3], Ishii [16], Ichihara and Ishii [14], and for problems with periodic boundary conditions, see for instance Mitake [20, 19, 21]. We refer also the reader to Ishii [15, 17] for an overview.

In the case of systems, we are interested in finding an ergodic constant vector $(c_1, \dots, c_m) \in \mathbb{R}^m$ and a function (v_1, \dots, v_m) such that

$$(1.8) \quad H_i(x, Dv_i) + \sum_{j=1}^m d_{ij}(x)v_j = c_i, \quad x \in \mathbb{T}^N, \quad i = 1, \dots, m$$

and, for all $i = 1, \dots, m$,

$$(1.9) \quad u_i(x, t) + c_i t \rightarrow v_i(x) \text{ uniformly as } t \text{ tends to infinity,}$$

where u is the solution of (1.1).

First results for the system (1.1) were obtained in [6] and [22]. More precisely, suppose that $H_i(x, p) = F_i(x, p) - f_i(x)$ satisfies the same properties as in [25] (see above and (1.39)). Assume that

$$(1.10) \quad \mathcal{F} = \{x_0 \in \mathbb{T}^N : f_i(x_0) = \min_{x \in \mathbb{T}^N} f_j(x) \text{ for all } i, j = 1, \dots, m\} \neq \emptyset,$$

i.e., the f_i 's attain their minimum at the same point with the same value. They prove that \mathcal{F} is a uniqueness set (which replaces \mathcal{F}_{scalar}). Then, the proof of the convergence is obtained.

It is based on the same ideas as in [25]. Let us outline the proof of convergence of [6]. To simplify, assume that

$$(1.11) \quad \sum_{j=1}^m d_{ij}(x) = 0, \quad \sum_{i=1}^m d_{ij}(x) = 0, \quad i, j = 1, \dots, m, \quad x \in \mathbb{T}^N.$$

Without loss of generality, we suppose that $f_i(x_0) = 0$ where $x_0 \in \mathcal{F}$ defined in (1.10). The coercivity of the Hamiltonians and the existence of a solution to the ergodic problem give the compactness of the sets $\{u_i(\cdot, t), t \geq 0\}$'s in $W^{1,\infty}(\mathbb{T}^N)$ (under the assumption (1.10) and $f_i(x_0) = 0$ for $x_0 \in \mathcal{F}$, we obtain $(c_1, \dots, c_m) = (0, \dots, 0)$). By summing the equations (1.1) for $i = 1, \dots, m$, one obtains

$$\sum_{i=1}^m \frac{\partial u_i}{\partial t} + \sum_{i=1}^m H_i(x, Du_i) + \sum_{i,j=1}^m d_{ij} u_j = 0.$$

Using (1.11) and $H_i(x, Du_i) \geq 0$ on \mathcal{F} , one gets easily that

$$(1.12) \quad \frac{\partial}{\partial t} \sum_{i=1}^m u_i(x, t) \leq 0 \quad \text{on } \mathcal{F}$$

and therefore $t \mapsto (u_1 + \dots + u_m)(\cdot, t)$ is nonincreasing and converges uniformly as $t \rightarrow +\infty$ on \mathcal{F} . From this fact, one can prove the convergence of *each* u_i on \mathcal{F} . Using that \mathcal{F} is a *uniqueness set*, (1.8) has a unique solution if one prescribes the values of v_i 's on (1.8). This is enough to conclude the convergence (1.9) in all \mathbb{T}^N .

The interest of studying this kind of systems relies on the nice control interpretation they have. They are related to optimal control of pathwise deterministic trajectories with random switching (see [6, Section 6] and the Appendix for details). When the f_i 's achieve their minimum at the same points with the same value, i.e., when $\mathcal{F} \neq \emptyset$, the interpretation of the convergence result in terms of control is clear: one should rather driving the trajectories to a common minimum of the f_i 's since these latters play the role of the running costs of the control problem.

The extension of such a result to the case $\mathcal{F} = \emptyset$, i.e, when the f_i 's do not have common minimum with the same value, was the most challenging issue which was addressed in [6] and one of the motivation of this paper. The following example shows the main difficulty we encounter.

Example 1.2.

$$(1.13) \quad \begin{cases} \frac{\partial u_1}{\partial t} + |Du_1|^2 + u_1 - u_2 = f_1(x), \\ \frac{\partial u_2}{\partial t} + |Du_2|^2 + u_2 - u_1 = f_2(x), \end{cases} \quad x \in \mathbb{R}.$$

In this example, we choose $f_1(x) = \cos^2 x + \sin x - \cos x$, $f_2(x) = \sin^2 x + \cos x - \sin x$. We have

$$\min f_1 = \min f_2 := m < 0 \text{ and } f_1(x) + f_2(x) = 1.$$

The last equality shows that f_1, f_2 do not attain their minima at the same point so $\mathcal{F} = \emptyset$.

Assume that $(w_1, w_2) \in C(\mathbb{R}) \times C(\mathbb{R})$ is a solution of

$$(1.14) \quad \begin{cases} |Dw_1|^2 + w_1 - w_2 = \cos^2 x + \sin x - \cos x, \\ |Dw_2|^2 + w_2 - w_1 = \sin^2 x + \cos x - \sin x, \end{cases} \quad x \in \mathbb{R}.$$

Such a solution clearly exists since we can take $(w_1, w_2) = (\sin(\cdot), \cos(\cdot))$. Then the ergodic constant in this example is 0.

Summing the equations of System (1.13), we have

$$\frac{\partial(u_1 + u_2)}{\partial t} + |Du_1|^2 + |Du_2|^2 = 1.$$

It follows that we cannot find any set where (1.12) holds and the proof of [6] and [22] does not work anymore.

However, and it is one of the main achievement of this paper, we will see below how our main result Theorem 1.4 can be applied to give a full answer to this problem, see Theorem 1.5. Moreover, we can improve directly the result of [22, 6], in a particular case, see Theorem 1.6. For a comparison of these two new results, see Section 1.5.

1.2. Main result. In this section, the existence of solutions to the stationary system (1.8) is made as an assumption and we will normalize the Hamiltonians so that the ergodic constant $(c_1, \dots, c_m) = (0, \dots, 0)$. So we introduce the assumptions on H_i (in general, the assumptions are made on $H_i - c_i$)

The main result of this work is the extension of Theorem 1.1 to the system (1.1). In order to explain the difficulty of this extension, we start by giving a rough idea of the proof of Theorem 1.1.

In [4], the authors first show that

$$(1.15) \quad \min_{x \in \mathbb{T}^N} \frac{\partial u}{\partial t}(x, t) \rightarrow 0 \text{ as } t \text{ tends to infinity.}$$

The main consequence of this fact is that the ω -limit set of $\{u(\cdot, t), t \geq 0\}$ contains only subsolutions of (1.4). This fact with the compactness of \mathbb{T}^N are enough to prove (1.5).

Now to prove (1.15), the authors perform a change of function of the form

$$(1.16) \quad \exp(w(x, t)) = u(x, t), \quad (x, t) \in \mathbb{T}^N \times (0, \infty).$$

Then w solves

$$(1.17) \quad \frac{\partial w}{\partial t} + F(x, w, Dw) = 0, \text{ with } F(x, w, p) = \exp(-w)H(x, \exp(w)p)$$

and F inherits the properties of H :

$$(1.18) \quad \begin{cases} \text{There exists a, possibly empty, compact set } \mathcal{K} \text{ of } \mathbb{T}^N \text{ such that} \\ \text{if } F(x, w, p) \geq \eta > 0 \text{ and } d(x, \mathcal{K}) \geq \eta, \text{ then } F_w(x, w, p) \geq \Psi(\eta) > 0 \text{ a.e.} \end{cases}$$

An application of the maximum principle yields that $t \mapsto \min_{x \in \mathbb{T}^N} \frac{\partial w}{\partial t}(x, t)$ is nonincreasing so it converges. If the limit is nonnegative, we obtain easily the convergence of $w(x, t)$ as $t \rightarrow \infty$. Otherwise, there exists some $\eta > 0$, $t_0 > 0$ such that for all $t \geq t_0$

$$(1.19) \quad \min_{x \in \mathbb{T}^N} \frac{\partial w}{\partial t}(x, t) \leq -\eta.$$

Set $z = \frac{\partial w}{\partial t}$ and $m(t) = \min_{x \in \mathbb{T}^N} z(x, t) := z(x_t, t)$. Differentiating (1.17) with respect to t , we obtain

$$\frac{\partial z}{\partial t} + F_w(x, w, t, Dw)z + F_p.Dz = 0.$$

Formally at x_t , we get

$$m' + F_w(x_t, w, t, Dw)m = 0.$$

Using (1.18) and (1.19), we get

$$m' + \Psi(\eta)m \geq 0, \text{ thus } m(t) \geq m(t_0)e^{-\Psi(\eta)(t-t_0)}$$

Letting t tends to infinity yields a contradiction with (1.19).

Now, let us turn to the case of systems through the typical example

$$\begin{cases} \frac{\partial u_1}{\partial t} + H_1(x, Du_1) + u_1 - u_2 = 0, \\ \frac{\partial u_2}{\partial t} + H_2(x, Du_2) + u_2 - u_1 = 0, \end{cases} \quad (x, t) \in \mathbb{T}^N \times (0, +\infty),$$

where the H_i 's satisfy (1.7). Similarly to the case of a single equation, if we perform the change of function $\exp(w_i) = u_i$, then we obtain the new system

$$(1.20) \quad \begin{cases} \frac{\partial w_1}{\partial t} + F_1(x, w_1, Dw_1) + 1 - \exp(w_2 - w_1) = 0, \\ \frac{\partial w_2}{\partial t} + F_2(x, w_2, Dw_2) + 1 - \exp(w_1 - w_2) = 0, \end{cases} \quad (x, t) \in \mathbb{T}^N \times (0, +\infty),$$

where $F_i(x, w, p) = \exp(-w)H_i(x, \exp(w)p)$. Let us try to mimic the above sketch of proof in the case of the system (1.20). We want to prove that

$$(1.21) \quad \min_{x \in \mathbb{T}^N, i=1,2} \frac{\partial w_i}{\partial t}(x, t) \rightarrow 0 \text{ as } t \text{ tends to infinity.}$$

An application of the maximum principle yields that $t \mapsto \min_{x \in \mathbb{T}^N, i=1,2} \frac{\partial w_i}{\partial t}(x, t)$ is non-increasing so it converges. If the limit is nonnegative, we obtain easily the convergence of $w_i(x, t)$ as $t \rightarrow \infty$. Otherwise, there exists some $\eta > 0$, $t_0 > 0$ such that for all $t \geq t_0$

$$(1.22) \quad \min_{x \in \mathbb{T}^N, i=1,2} \frac{\partial w_i}{\partial t}(x, t) \leq -\eta.$$

Set $z_i = \frac{\partial w_i}{\partial t}$ and assume that

$$(1.23) \quad m(t) = \min_{x \in \mathbb{T}^N, i=1,2} z_i(x, t) := z_1(x, t).$$

This fact, (1.20) and (1.22) only give

$$(1.24) \quad F_1(x_t, w_1, Dw_1) + 1 - \exp(w_2 - w_1)(x_t, t) \geq \eta.$$

We see that (1.18) cannot apply here, since we cannot control the additional term $1 - \exp(w_2 - w_1)(x_t, t)$ using only the information given by (1.23).

It forces us to modify the assumptions of Theorem 1.1 to be able to treat this difficulty. The new assumption should encompass typical Hamiltonians which satisfy the assumptions of Theorem 1.1. We propose two sets of assumptions.

The following set of assumptions seems to be the natural extension of (1.7) to systems. We assume for $i = 1, \dots, m$ that

$$(1.25) \quad \begin{cases} (i) \text{ The function } p \mapsto H_i(x, p) \text{ is differentiable a.e. in } x \in \mathbb{T}^N. \\ (ii) (H_i)_p p - H_i \geq 0 \text{ for a.e. } (x, p) \in \mathbb{T}^N \times \mathbb{R}^N, \\ (iii) \text{ There exists a, possibly empty, compact set } K \text{ of } \mathbb{T}^N \text{ such that} \\ \quad (a) H_i(x, p) \geq 0 \text{ on } K \times \mathbb{R}^N, \\ \quad (b) \text{ If } H_i(x, p) \geq \eta > 0 \text{ and } d(x, K) \geq \eta, \text{ then } (H_i)_p p - H_i \geq \Psi(\eta) > 0. \end{cases}$$

Comparing (1.25) in the case of a single equation ($m = 1$) and (1.7), we see that (1.25) (ii) is the only additional assumption, which is crucial in the proof of (1.29) (see the key Lemma 6.2).

Here is another set of assumptions which is not covered by (1.25).

$$(1.26) \quad \left\{ \begin{array}{l} (i) \text{ The function } p \mapsto H_i(x, p) \text{ is differentiable a.e. in } x \in \mathbb{T}^N \text{ for all } i. \\ (ii) (H_i)_p p - H_i \geq 0 \text{ for a.e. } (x, p) \in \mathbb{T}^N \times \mathbb{R}^N \text{ for all } i. \\ (iii) H_1(x, p) = \max_{1 \leq i \leq m} H_i(x, p) \text{ for } (x, p) \in \mathbb{T}^N \times \mathbb{R}^N. \\ (iv) \text{ There exists a, possibly empty, compact set } K \text{ of } \mathbb{T}^N \text{ such that} \\ \quad (a) H_i(x, p) \geq 0 \text{ on } K \times \mathbb{R}^N \text{ for all } i, \\ \quad (b) \text{ If } H_1(x, p) \geq \eta > 0 \text{ and } d(x, K) \geq \eta, \text{ then } (H_1)_p p - H_1 \geq \Psi(\eta) > 0. \end{array} \right.$$

Condition (1.26)(iii) means that we require the existence of a biggest Hamiltonian and Condition (1.7)(ii) has to be satisfied only for this biggest Hamiltonian H_1 . Roughly speaking, everything is under the control of H_1 .

Let us roughly explain how we can overcome the difficulty mentioned after (1.24), by using (1.25) or (1.26). As for single equations, we make the change of function $\exp(w_i) = u_i$. The function w is solution to the new system

$$(1.27) \quad \frac{\partial w_i}{\partial t} + F_i(x, w_i, Dw_i) + \sum_{j=1}^m d_{ij} \exp(w_j - w_i) = 0, \quad i = 1, \dots, m,$$

with $F_i(x, w, p) = \exp(-w) H_i(x, \exp(w)p)$. Without loss of generality, we can replace w with $v \in W^{1,\infty}(\mathbb{T}^N \times (0, \infty))^m$ belonging to the ω -limit set of $\{w(\cdot, \cdot + t) + ct, t \geq 0\}$, where c is the ergodic vector. For shortness, we still denote v by w .

Under (1.25), the properties of the F_i 's which are inherited from the H_i 's are

$$(1.28) \quad \left\{ \begin{array}{l} (i) (F_i)_w(x, w, p) \geq 0 \text{ for a.e. } (x, w, p), \\ (ii) \text{ There exists a, possibly empty, compact set } \mathcal{K} \text{ of } \mathbb{T}^N \text{ such that} \\ \quad \text{if } F_i(x, w, p) \geq \eta > 0 \text{ and } d(x, \mathcal{K}) \geq \eta, \text{ then } (F_i)_w(x, w, p) \geq \Psi(\eta) > 0 \text{ a.e.} \end{array} \right.$$

Surprisingly, in this case, we are able to reinforce (1.23) by choosing a $x_t \in \mathbb{T}^N$ such that

$$(1.29) \quad m(t) = \min_{x \in \mathbb{T}^N, i=1,2} z_i(x, t) := z_1(x_t, t) = z_2(x_t, t).$$

This fact, (1.20) and (1.22) give us *two* inequalities

$$\left\{ \begin{array}{l} F_1(x_t, w_1, Dw_1) + 1 - \exp(w_2 - w_1)(x_t, t) \geq \eta, \\ F_2(x_t, w_2, Dw_2) + 1 - \exp(w_1 - w_2)(x_t, t) \geq \eta. \end{array} \right.$$

Then, if $w_1(x_t, t) \leq w_2(x_t, t)$ for instance, we have

$$(1.30) \quad F_1(x_t, w_1, Dw_1) \geq \eta.$$

Now, we can apply assumption (1.28)(ii) and continue the proof accordingly. The fact that the minimum in (1.29) is achieved at the same point both for z_1 and z_2 is a crucial point. This is a consequence of our new assumption (1.25)(ii).

Now, in the case of (1.26), the properties of the F_i 's which are inherited from the H_i 's are

$$(1.31) \quad \begin{cases} (i) & (F_i)_w(x, w, p) \geq 0 \text{ for a.e. } (x, w, p), \\ (ii) & w_1(x, t) \leq w_i(x, t) \text{ for all } (x, t) \in \mathbb{T}^N \times (0, \infty) \text{ and } i = 1, \dots, m. \\ (iii) & \text{There exists a, possibly empty, compact set } \mathcal{K} \text{ of } \mathbb{T}^N \text{ such that} \\ & \quad \text{if } F_1(x, w, p) \geq \eta > 0 \text{ and } d(x, \mathcal{K}) \geq \eta, \text{ then } (F_1)_w(x, w, p) \geq \Psi(\eta) > 0 \text{ a.e..} \end{cases}$$

Let us mention that (1.31)(ii) is a consequence of the new assumption (1.26)(iii). In this case, we replace (1.29) by

$$m_1(t) = \min_{x \in \mathbb{T}^N} z_1(x, t) := z_1(x_t, t).$$

Here, (1.31)(ii) implies easily (1.30), so we can apply (1.31) (ii) to prove that $m_1(t)$ tends to 0 as $t \rightarrow \infty$ and this is enough to the convergence (1.21).

To state our result, we need to introduce some assumptions on the coupling. The irreducibility of the coupling is a classical assumption when dealing with systems of PDEs. Roughly speaking, when the coupling is irreducible, it means that the system is not separated into many smaller systems.

$$(1.32) \quad \begin{aligned} & \text{We say that } D(\cdot) \text{ is irreducible if for any } x \in \mathbb{T}^N \text{ and subset } \mathcal{I} \subsetneq \{1, \dots, m\}, \\ & \text{there exist } i \in \mathcal{I} \text{ and } j \notin \mathcal{I} \text{ such that } d_{ij}(x) \neq 0. \end{aligned}$$

Under (1.25), we need the existence of a line with nonzero coefficients in the coupling, that is,

$$(1.33) \quad \text{For all } x \in \mathbb{T}^N, \text{ there exists } i \text{ such that } d_{ij}(x) \neq 0, \text{ for all } j = 1, \dots, m.$$

Under (1.26), we need the following

$$(1.34) \quad \begin{aligned} & \text{We say that } D(\cdot) \text{ is nonzero if for any } x \in \mathbb{T}^N, i, j = 1, \dots, m, \text{ there exists} \\ & k \in \{1, \dots, m\} \text{ such that } d_{ik}(x) \neq 0 \text{ and } d_{jk}(x) \neq 0. \end{aligned}$$

Let us comment on this last condition. The main consequence of systems which have a biggest Hamiltonians, i.e., for which (1.26)(iii) holds, is

Lemma 1.3. *Consider (1.1) where the coupling $D(x)$ satisfies (1.34). Assume moreover that (1.26)(iii) holds. Then we have*

$$\limsup_{t \rightarrow \infty} \max_{x \in \mathbb{T}^N, j=1, \dots, m} (u_1(x, t) - u_j(x, t)) \leq 0.$$

To prove this result, we subtract the 1st equation from the j th in (1.1) and use (1.34). Actually, (1.34) is not the most general assumption we can state but maybe the simplest one.

We are now able to state our main result, the proof of which is given in Section 6

Theorem 1.4. (Main convergence result) *Suppose that the coupling D is independent of x and one of the two following conditions holds*

- (i) H_i satisfies (1.25) and (1.33) holds,
- (ii) H_i satisfies (1.26) and (1.32), (1.34) hold.

Then, the solution $u = (u_1, \dots, u_m) \in W^{1, \infty}(\mathbb{T}^N \times (0, \infty))^m$ of (1.1) converges uniformly to a solution $(v_{\infty 1}, \dots, v_{\infty m})$ of (1.8).

The proof of this theorem is difficult, very technical and relies on several auxiliary results.

1.3. Some applications of the main result. First important examples for which the main result can be applied are nonconvex systems. The following example is drawn from [4],

$$(1.35) \quad H_i(x, p) = \psi_i(x, p) F_i(x, \frac{p}{|p|}) - f_i(x),$$

where $f_i \in C(\mathbb{T}^N)$ is nonnegative, $F_i \in C(\mathbb{T}^N \times \mathbb{R}^N \setminus \{0\})$ is strictly positive and $\psi_i(x, p) = |p + q_i(x)|^2 - |q_i(x)|^2$. Moreover, we assume that

$$(1.36) \quad K = \{x \in \mathbb{T}^N : f_i(x) = |q_j(x)| = 0 \text{ for all } i, j = 1, \dots, m\} \neq \emptyset.$$

Under the above conditions, we have $c = (0, \dots, 0)$. We compute

$$(1.37) \quad (H_i)_p(x, p) p - H_i(x, p) = |p|^2 F_i(x, \frac{p}{|p|}) + f_i(x).$$

It is straightforward to see that H_i satisfies (1.25) with K defined as in (1.36).

Another application is the system

$$(1.38) \quad \begin{cases} \frac{\partial u_1}{\partial t} + |Du_1 + f_1(x)|^2 - |f_1(x)|^2 + u_1 - u_2 = 0, \\ \frac{\partial u_2}{\partial t} + |Du_2 + f_2(x)|^2 - |f_2(x)|^2 + u_2 - u_1 = 0. \end{cases} \quad (x, t) \in \mathbb{T}^N \times (0, +\infty),$$

where $f_i \in C(\mathbb{T}^N)$. This example was neither covered by previous convergence results nor by the other convergence results described below in Section 1.4. It solves a question which was asked in [6] where the authors obtained the convergence for (1.38) with f_1, f_2 are constant. It is straightforward to check that (1.25) holds but it is also a particular case of the more general application we give now.

The following theorem is an important consequence of the main result in the case of systems with strictly convex Hamiltonians. To enlight the main ideas, we provide the proof hereunder in the particular case where there exists a C^1 subsolution to the ergodic problem. The proof for the general case is provided in Section 7.

Theorem 1.5. *Suppose that D is independent of x and satisfies (1.32) and (1.33). We assume that, for $i = 1, \dots, m$,*

$$H_i(x, \cdot) \text{ is strictly convex and coercive uniformly in } x \in \mathbb{T}^N.$$

Then there exists $c = (c_1, \dots, c_1)$ and a solution $v \in W^{1,\infty}(\mathbb{T}^N)^m$ of (1.8) such that $u + ct \rightarrow v$ in $C(\mathbb{T}^N)^m$, where u is the solution of (1.1).

This theorem extends the result of [8] to systems. It also gives a full answer to the Eikonal type Hamiltonians case of [6, 22]: when the Hamiltonians are strictly convex, one has the convergence regardless \mathcal{F} is empty or not. In particular we obtain the convergence in the case of Example 1.2. In the non-strictly convex case, it is known that we cannot hope to obtain a convergence result, see Example 1.8 for a counter-example. In the next section, we present a particular result in this context with not necessarily strictly convex Hamiltonians.

We learnt very recently that Mitake and Tran [24] obtained the same result, Theorem 1.5, by a different approach. They use a dynamical approach which corresponds, in the case of systems, to the method of [8]. Here, the result is a particular case of a general PDE approach.

Proof of Theorem 1.5 (when there exists a C^1 subsolution to (1.8)). To make the presentation easier in this introduction, we consider a particular case containing the main ideas of the proof. See Section 7 for the general case. We then assume that v is a C^1 subsolution of (1.8).

Set $w_i = u_i - v_i$ for $i = 1, \dots, m$ where u is the solution of (1.1). Then w is the bounded solution of

$$\frac{\partial w_i}{\partial t} + H_i(x, Dv_i + Dw_i) - H_i(x, Dv_i) - g_i(x) + \sum_{j=1}^m d_{ij}(x)w_j = 0, \quad i = 1, \dots, m.$$

Where $g_i(x) := -H_i(x, Dv_i) - \sum_{j=1}^m d_{ij}(x)v_j \geq 0$ and $g_i \in C(\mathbb{T}^N)$ for all $i = 1, \dots, m$ since v is a C^1 subsolution of (1.8). We introduce the new Hamiltonians $G_i(x, p) = H_i(x, p + Dv_i) - H_i(x, Dv_i) - g_i(x)$. Since H_i is coercive then the solutions u of (1.1) and v of (1.8) are Lipschitz continuous. In order to apply Theorem 1.4, it is then sufficient to check that (1.25) holds with $K = \emptyset$ and p bounded. It is left to the reader. Finally, we obtain the large time behavior of the solution by applying Theorem 1.4. \square

1.4. Miscellaneous results. We obtain some particular results under different assumptions on the Hamiltonians. These results are not completely covered by the main result and bring to light some interesting phenomena.

1.4.1. Hamiltonians of Eikonal type. We focus on the setting of Namah and Roquejoffre [25], i.e., when the Hamiltonians take the form

$$H_i(x, p) = F_i(x, p) - f_i(x), \quad x \in \mathbb{T}^N, p \in \mathbb{R}^N.$$

For $i = 1, \dots, m$, we assume

$$(1.39) \quad F_i(x, \cdot) \text{ is convex, coercive uniformly for } x \in \mathbb{T}^N \text{ and } F_i(x, p) \geq F_i(x, 0) = 0.$$

We can extend the results of [22, 6] when \mathcal{F} defined by (1.10) is replaced by

$$(1.40) \quad \mathcal{S} := \{x_0 \in \mathbb{T}^N, f_i(x_0) = \min_{x \in \mathbb{T}^N} f_i(x), \text{ for all } i\} \neq \emptyset.$$

This latter condition means that the f_i 's attain their minimum at the same point but *their value at this point may be different*.

Under this new weaker condition, we can prove the result when the coupling matrix is independent of x . The idea is that we can find an explicit formula for the c_i 's together with a constant solution of the ergodic problem. It is then possible to bring the problem back to the one proved in [6] and [22].

Theorem 1.6. *Assume that D is independent of x and satisfies (1.32). Moreover, we assume that $F_i \in C(\mathbb{T}^N \times \mathbb{R}^N)$, $f_i \in C(\mathbb{T}^N)$ satisfy (1.39) and (1.40), respectively. Then there exist $c = (c_1, \dots, c_m) \in \mathbb{R}^m$ and $u_\infty \in W^{1,\infty}(\mathbb{T}^N)$ solution of (1.8) such that $u + ct \rightarrow u_\infty$ in $C(\mathbb{T}^N)^m$, where u is the solution of (1.1).*

1.4.2. Strictly convex Hamiltonians. Theorem 1.5 is a natural extension of Davini-Siconolfi [8] to systems. Here is another result in this direction with the following assumptions.

$$(1.41) \quad H_i(x, p) \geq \rho(x)K_i(p), \quad H_i(x, 0) = K_i(0) = 0, \quad K_i \text{ is convex for } i = 1, \dots, m,$$

where ρ is a positive \mathbb{T}^N -periodic function.

$$(1.42) \quad \text{There exists at least one } i \in \{1, \dots, m\} \text{ such that } K_i \text{ is strictly convex.}$$

Our result is the following

Theorem 1.7. *Assume that H_i satisfies (1.41)-(1.42) and the coupling D is independent of x and satisfies (1.32). Assume $u \in W^{1,\infty}(\mathbb{T}^N \times (0, \infty))^m$ is the solution of (1.1). Then, there exists $U \in \mathbb{R}$ such that $u_i \rightarrow U$ uniformly for all i .*

Typical Hamiltonians satisfying Theorem 1.7 are given in Example 4.1. If we compare our result for systems with the corresponding one in [8] for a single equation, we see that we need the additional assumption (1.41). However, we can prove the convergence result for systems with convex Hamiltonians, when *merely one of the H_i 's in (1.1) is strictly convex*. It means that one strictly convex Hamiltonian is enough to control the large time behavior of the whole system. This condition is also necessary, see the counter-example 1.8 when all the H_i 's are convex but no one is strictly convex. Theorem 1.7 improves a similar result of [22] which was obtained for (1.43) in the case $\alpha = \beta = 2$.

Example 1.8. Consider

$$(1.43) \quad \begin{cases} \frac{\partial u_1}{\partial t} + |Du_1 + 1|^\alpha - 1 + u_1 - u_2 = 0, \\ \frac{\partial u_2}{\partial t} + |Du_2 + 1|^\beta - 1 - u_1 + u_2 = 0, \\ u_1(x, 0) = u_2(x, 0) = \sin(x). \end{cases} \quad (x, t) \in \mathbb{R} \times (0, +\infty),$$

If $\alpha = \beta = 1$, then (1.41) holds and the unique solution of (1.43) is $u(x, t) = (\sin(x-t), \sin(x-t))$ which clearly does not converge as $t \rightarrow \infty$. This is a counterexample for which we do not have the convergence when (1.41) holds alone. If $\alpha \geq 1$, $\beta > 1$, then (1.41)-(1.42) hold and we can prove the convergence of the solution for this system.

1.4.3. The case when all Hamiltonians are identical. We focus on the case where all the Hamiltonians in the system (1.1) are identical. In this setting, we can give an easy proof of the large time behavior of solutions of (1.1) for a very wide class of Hamiltonians. We show that the large time behavior for systems is actually *inherited directly from the case of a single equation*, i.e., we prove that the convergence result holds for systems as soon as it holds for the corresponding single equation. This result generalizes a result of [22] obtained for systems under assumptions related to the framework of [4]. Here, we prove the result under a very natural assumption when studying the large time behavior for system.

(1.44) The solution of equation (1.3) has large time behavior .

The result is

Theorem 1.9. *Call $u \in BUC(\mathbb{T}^N \times (0, \infty))^m$ the solution of (1.1). Assume that the coupling $D(x)$ satisfies (1.34), $H_i = H_j$ for all $i, j = 1, \dots, m$ and (1.44) holds. Then $u(\cdot, t)$ converges uniformly to a solution v of (1.4).*

1.5. The relation among all convergence results. Let us compare all results which have been stated above to give the reader a clearer overview. For Hamiltonians of Eikonal type, we obtain, according to us, a satisfactory answer to the general case for systems which was left open in [6] and [22], see Theorem 1.5. When the Hamiltonians are not *strictly* convex, we cannot hope a convergence result in the whole generality but, in this case too, we obtain a new result, Theorem 1.6. The important point in Theorem 1.6 is that we can compute the exact value of the ergodic constant. Actually, when we have in hand this ergodic constant, we can prove that the assumptions of Theorem 1.4 hold. So Theorem 1.6 is contained in the main result Theorem 1.4 but let us underline again that the important part in the proof of Theorem 1.6 is the computation of the ergodic constant, see Lemma 3.1.

A consequence of Theorem 1.4 is Theorem 1.5 which proves the convergence result for systems of strictly convex Hamiltonians. This is a natural extension of the result in [8] to systems. We also obtain another result for strictly convex Hamiltonians, see Theorem 1.7. This latter result is not completely covered by Theorem 1.4 since we need only one *strictly* convex Hamiltonian in Theorem 1.7. We refer the reader to Example 1.8 with $\alpha \geq 1$, $\beta > 1$ and Example 5.2 for more discussions.

Finally, Theorem 1.9 requires all the Hamiltonians to be identical. This result is not contained in Theorem 1.4 since it works for a very wide class of Hamiltonians provided the convergence result holds for the corresponding single equation.

1.6. Organization of the paper. This paper is organized as follows. In Section 2, preliminary results for the coupling of the systems are given, the ergodic problem for coercive Hamiltonians is solved and basic properties of the solutions like existence, uniqueness, Lipschitz regularity and relative compactness are presented. Next sections are devoted to the proofs of the theorems stated in the introduction. The proof of Theorem 1.6 is given in Section 3, the proof of Theorem 1.7 is given in Section 4 and the proof of Theorem 1.9 is given in Section 5. Section 6 contains the proof of the main result (Theorem 1.4). This is the most technical and involved part. Finally, we give the proof of Theorem 1.5 in Section 7. Since the ideas of this proof are based on the ideas used in the proof Theorem 1.4, we strongly recommend the reader to read it after reading Section 6.

Notations: Since we only work with viscosity solutions in this paper, we will drop the term “viscosity” hereafter. We denote by $C(\mathbb{T}^N)^m$ the set of functions $u = (u_1, \dots, u_m)$, where $u_i : \mathbb{T}^N \rightarrow \mathbb{R}$ is continuous for all $i = 1, \dots, m$.

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2. SOME PRELIMINARY RESULTS

2.1. Preliminaires on coupling matrices. Here is one key property of irreducible matrices.

Lemma 2.1. ([6]) *Suppose that $D(x) = (d_{ij}(x))_{1 \leq i, j \leq m}$ satisfies (1.2) and (1.32). Then for all $x \in \mathbb{T}^N$, $D(x)$ is degenerate of rank $m - 1$, $\ker(D(x)) = \text{span}\{(1, \dots, 1)\}$ and the real part of each nonzero complex eigenvalue of $D(x)$ is positive. Moreover there exists a positive continuous function $\Lambda = (\Lambda_1, \dots, \Lambda_m) : \mathbb{T}^N \rightarrow \mathbb{R}^m$ such that $\Lambda(x) > 0$ and $D(x)^T \Lambda(x) = 0$ for all $x \in \mathbb{T}^N$.*

2.2. Ergodic problem for coercive Hamilton-Jacobi systems. We solve the ergodic problem for systems of Hamilton-Jacobi equations with coercive Hamiltonians.

Theorem 2.2. *Consider system (1.1) where H_i is coercive in p uniformly in x , $D(\cdot)$ satisfies (1.32). Then, there is a solution $((c_1, \dots, c_1), v) \in \mathbb{R}^m \times W^{1, \infty}(\mathbb{T}^N)^m$ to (1.8) with $(c_1, \dots, c_1) \in \ker D(x)$ for all $x \in \mathbb{T}^N$. Moreover, (c_1, \dots, c_1) is unique in $\ker D$.*

Remark 2.3. The classical result for scalar first-order Hamilton-Jacobi equations is due to Lions-Papanicolaou-Varadhan [18]. In general, the ergodic problem (1.4) is solved in the following way: we first prove a gradient bound for the regularized equation $\lambda v^\lambda + H(x, Dv^\lambda) = 0$. Due to the coercivity of H , this gradient bound is independent of λ . Since v^λ may not

be bounded, we make a change of function $w^\lambda = v^\lambda - v^\lambda(x_0)$ in the equation. It follows $|Dw^\lambda| \leq L$ and w^λ is uniformly bounded thanks to the compactness of \mathbb{T}^N . And then, the requirements of Ascoli's theorem are fulfilled. Here, for systems with a x -dependent coupling matrix, such an approach does not work since the change of variable $w_i^\lambda = v_i^\lambda - v_i^\lambda(x_0)$ leads to additional terms in the system which are difficult to control. That is why we required the coupling matrix to be independent of x to prove [6, Theorem 4.3]. Here we can overcome this difficulty with Lemma 2.5 below. Let us point out that we cannot use the gradient bound to obtain (2.3).

Proof of Theorem 2.2. Step 1. Ergodic approximation. We consider the ergodic approximation to (1.8): for $\lambda \in (0, 1)$, let $v^\lambda = (v_1^\lambda, \dots, v_m^\lambda)$ be the solution of

$$(2.1) \quad \lambda v_i + H_i(x, Dv_i) + \sum_{j=1}^m d_{ij}(x)v_j = 0 \quad x \in \mathbb{T}^N, \quad 1 \leq i \leq m.$$

Lemma 2.4. ([6]) *There exist a unique solution v^λ of (2.1) and a constant $M > 0$ independent of λ such that*

$$(2.2) \quad 0 \leq v_i^\lambda \leq \frac{M}{\lambda} \quad \text{and} \quad \|Dv_i^\lambda(\cdot)\|_\infty \leq M, \quad i = 1, \dots, m.$$

Step 2. Some uniform bounds. Here is the key lemma which helps us improve the result proved in [6].

Lemma 2.5. *Under the assumptions of Theorem 2.2, there exists a constant M such that for all $x \in \mathbb{T}^N$, $i = 1, \dots, m$, we have*

$$(2.3) \quad |v_i^\lambda(x) - v_1^\lambda(x^*)| \leq M \quad \text{for any fixed } x^* \in \mathbb{T}^N.$$

The proof is postponed.

From Ascoli's theorem, there exist $c = (c_1, \dots, c_m) \in \mathbb{R}^m$ and $v \in C(\mathbb{T}^N)^m$ such that, up to subsequences, for $i = 1, \dots, m$

$$\lambda v_i^\lambda(x^*) \rightarrow -c_i \quad \text{and} \quad v_i^\lambda - v_1^\lambda(x^*) \rightarrow v_i \quad \text{in } C(\mathbb{T}^N) \quad \text{as } \lambda \rightarrow 0.$$

Notice that c_i does not depend on the choice of x^* since, for any $x^*, y^* \in \mathbb{T}^N$

$$(2.4) \quad |-\lambda v_i^\lambda(x^*) + \lambda v_i^\lambda(y^*)| \leq \lambda M |x^* - y^*| \rightarrow 0 \quad \text{as } \lambda \rightarrow 0.$$

Moreover, multiplying (2.1) by λ for all i and sending $\lambda \rightarrow 0$, we obtain $-\sum_j d_{ij}(x)c_i = 0$ which gives $D(x)c = 0$ and therefore $c \in \ker D(x)$ for all $x \in \mathbb{T}^N$.

Step 3. Stability result for viscosity solutions and conclusion. We rewrite (2.1) as

$$(2.5) \quad \lambda v_i^\lambda + H_i(x, D(v_i^\lambda - v_1^\lambda(x^*))) + \sum_{j=1}^m d_{ij}(v_j^\lambda - v_1^\lambda(x^*)) = 0 \quad \text{in } \mathbb{T}^N,$$

by noting that $\sum_{j=1}^m d_{ij}(x)v_1^\lambda(x^*) = 0$ for all $x \in \mathbb{T}^N$ thanks to (1.2).

We then use the stability result and pass to the limit in (2.5) to get

$$(2.6) \quad H_i(x, Dv_i) + \sum_{j=1}^m d_{ij}v_j(x) = c_i \quad \text{in } \mathbb{T}^N, \quad 1 \leq i \leq m.$$

Then $(c, v(\cdot))$ is solution to (1.8). The function v depends on x^* but c does not.

From Lemma 2.1, the kernel of D is spanned by $(1, \dots, 1)$. Thus, any $c \in \ker D$ has the form (c_1, \dots, c_1) . The proof of uniqueness of c is classical and can be found in [6]. \square

Proof of Lemma 2.5. We set

$$w_i(x) = w_i^\lambda(x) = v_i^\lambda(x) - v_1^\lambda(x^*).$$

Thanks to (2.2), obviously $|w_1| \leq M$. Using (2.2) again for (2.5), we have

$$\left| \sum_{j=2}^m d_{ij}(x) w_j(x) \right| \leq C \text{ for all } i = 1, \dots, m \text{ and } x \in \mathbb{T}^N,$$

where C is independent of x and λ . For any $i \geq 2$, we have

$$d_{ii}|w_i| = \left| \sum_{j=2}^m d_{ij} w_j - \sum_{j=2, j \neq i}^m d_{ij} w_j \right| \leq C + \left| \sum_{j=2, j \neq i}^m d_{ij} w_j \right| \leq C - \sum_{j=2, j \neq i}^m d_{ij} |w_j|,$$

i.e.,

$$(2.7) \quad \sum_{j=2}^m d_{ij} |w_j| \leq C \text{ for any } i \geq 2.$$

Call $(\Lambda_1, \dots, \Lambda_m)$ the function given by Lemma 2.1. We have

$$-\sum_{j=2}^m |w_j| d_{1j} \Lambda_1 = \sum_{j=2}^m |w_j| \left(\sum_{i=2}^m d_{ij} \Lambda_i \right) = \sum_{i=2}^m \Lambda_i \sum_{j=2}^m d_{ij} |w_j| \leq C \left(\sum_{i=2}^m \Lambda_i \right).$$

Assume $|w_2| = \min_{j \geq 2} |w_j|$, we have

$$d_{11}|w_2| = -\sum_{j=2}^m d_{1j}|w_2| \leq -\sum_{j=2}^m |w_j| d_{1j} \leq C \frac{\sum_{i=2}^m \Lambda_i}{\Lambda_1}.$$

Thanks to (1.32), we have $d_{11} > 0$. Using the compactness of \mathbb{T}^N and continuity of the coupling, there exists $\delta_0 > 0$ such that $d_{11}(x) \geq \delta_0$ for all $x \in \mathbb{T}^N$. Therefore, we have

$$|w_2| \leq C \frac{\sum_{i=2}^m \Lambda_i}{\Lambda_1 \delta_0}.$$

We finish the proof by a reduction argument, i.e., we assume that

$$\sum_{j=k}^m d_{ij} |w_j| \leq C \text{ for any } 3 \leq k \leq m \text{ and } |w_l| \leq C \text{ for } 1 \leq l \leq k-1,$$

and we will show that $|w_k| \leq C'$. By similar arguments like those to obtain the bound for $|w_2|$, we then assume that $|w_k| = \min_{j \geq k} |w_j|$. We have

$$\left(\sum_{i=1}^{k-1} \Lambda_i \sum_{j=1}^{k-1} d_{ij} \right) |w_k| = \left(-\sum_{j=k}^m \sum_{i=1}^{k-1} d_{ij} \Lambda_i \right) |w_k| \leq -\sum_{j=k}^m |w_j| \sum_{i=1}^{k-1} d_{ij} \Lambda_i \leq C \left(\sum_{i=k}^m \Lambda_i \right).$$

If $\sum_{i=1}^{k-1} \Lambda_i \sum_{j=1}^{k-1} d_{ij}(x) > 0$ for all $x \in \mathbb{T}^N$, the conclusion follows easily by the compactness of \mathbb{T}^N and the continuity of the coupling. We then assume by contradiction that $\sum_{i=1}^{k-1} \Lambda_i \sum_{j=1}^{k-1} d_{ij}(x_0) = 0$ for some $x_0 \in \mathbb{T}^N$, (1.2) yields $d_{ij}(x_0) = 0$ for all $1 \leq i \leq k-1$, $k \leq j \leq m$. We get a contradiction with the choice $\mathcal{I} = \{1, \dots, k-1\}$ in (1.32). \square

2.3. Maximum principle and compactness properties of the solution.

Proposition 2.6. *Suppose that u, v are a subsolution and a supersolution of (1.1), respectively. Assume that either the H_i 's are coercive in p or u (or v) is Lipschitz.*

(i) *Let $u_0, v_0 \in C(\mathbb{T}^N)^m$. If u, v are respectively a subsolution and a supersolution of (1.1), then for any $t \geq 0$,*

$$(2.8) \quad \max_{1 \leq i \leq m} \sup_{\mathbb{T}^N} (u_i(\cdot, t) - v_i(\cdot, t)) \leq \max_{1 \leq i \leq m} \sup_{\mathbb{T}^N} (u_i(\cdot, 0) - v_i(\cdot, 0))^+.$$

(ii) *For any $u_0 \in C(\mathbb{T}^N)^m$, there exists a unique continuous solution of (1.1).*

Using the existence of solutions of the ergodic problem proved in Theorem 2.2, we can prove

Proposition 2.7. ([6]) *Under the assumptions of Theorem 2.2, let $u_0 \in W^{1,\infty}(\mathbb{T}^N)^m$ and u be the solution of (1.1) with initial data u_0 . Then*

$$\begin{aligned} |u(x, t) + ct| &\leq C, & x \in \mathbb{T}^N, t \in [0, \infty), \\ |u(x, t) - u(y, s)| &\leq L(|x - y| + |t - s|), & x, y \in \mathbb{T}^N, t, s \in [0, \infty), \end{aligned}$$

with C, L are constant.

From Proposition 2.7 and Ascoli theorem, we obtain easily the relative compactness of $\{u(\cdot, \cdot + t) + ct, t \geq 0\}$ in $C(\mathbb{T}^N)$.

2.4. First partial convergence result. The following result will be often used in the sequel. Roughly speaking, it gives the convergence of the solution on the set where the Hamiltonians are nonnegative.

Lemma 2.8. ([6]) *Let $u = (u_1, \dots, u_m)$ be a solution of (1.1) with $c = (0, \dots, 0)$. Assume that (1.1) admits a maximum principle and there exists a nonempty compact set K of \mathbb{T}^N such that*

$$H_i(x, p) \geq 0 \text{ on } K \times \mathbb{R}^N \text{ for all } i = 1, \dots, m.$$

Assume one of the two following conditions holds

- (i) (1.33) holds and $u \in BUC(\mathbb{T}^N \times (0, \infty))^m$,
- (ii) (1.32) holds and either H is coercive in p uniformly in x or $u \in W^{1,\infty}(\mathbb{T}^N \times (0, \infty))^m$.

Then $u(\cdot, t)$ converge as t tends to infinity for all $x \in K$.

Proof of Lemma 2.8. The proof for the case (i) holds is given in the step 3 of the proof of Lemma 6.5. We only outline the proof when (iii) holds, see [6, Lemma 5.6] for details. Formally, since $H_i \geq 0$ and $D(x)^T \Lambda(x) = 0$ for $x \in K$ where Λ is given by Lemma 2.1, we have

$$(2.9) \quad \frac{\partial(\sum_{i=1}^m \Lambda_i u_i)}{\partial t} \leq 0, \quad (x, t) \in K \times (0, \infty).$$

So for $x \in K$, $t \mapsto \sum_{i=1}^m \Lambda_i(x) u_i(x, t)$ is nondecreasing and therefore, it converges to some function $g(x)$ as t tends to infinity.

In (ii), if $u \in W^{1,\infty}(\mathbb{T}^N \times (0, \infty))^m$, then we can see easily that the conclusion of Proposition 2.7 holds. Otherwise if the H_i 's are coercive, by applying Proposition 2.7, then we still have $(u(\cdot, t))_{t \geq 0}$ is relatively compact in $C(\mathbb{T}^N)$ and there exists a sequence $t_n \rightarrow +\infty$ such

that $u(\cdot, t_n)$ converges uniformly on \mathbb{T}^N as $n \rightarrow +\infty$. From the maximum principle for system (1.1), we obtain that for all $n, q \in \mathbb{N}$,

$$\max_{1 \leq i \leq m} \sup_{\mathbb{T}^N \times [0, +\infty)} |u_i(\cdot, t_n + \cdot) - u_i(\cdot, t_q + \cdot)| \leq \max_{1 \leq i \leq m} \sup_{\mathbb{T}^N} |u_i(\cdot, t_n) - u_i(\cdot, t_q)|$$

and therefore $(u(\cdot, t_n + \cdot))_n$ is a Cauchy sequence in $W^{1,\infty}(\mathbb{T}^N \times [0, +\infty))$. Thus it converges uniformly to some function $w \in W^{1,\infty}(\mathbb{T}^N \times [0, +\infty))$. By the stability result for viscosity solutions, w is still a solution of (1.1) (see [1, 2, 9] for details). Since $\sum_{i=1}^m \Lambda_i v_i(x_0, t)$ converges, $\sum_{i=1}^m \Lambda_i w_i(x_0, t)$ does not depend on t anymore for $x_0 \in K$. It follows

$$\frac{\partial(\sum_{i=1}^m \Lambda_i w_i)(x_0, t)}{\partial t} = 0, \quad x_0 \in K.$$

This implies

$$\sum_{i=1}^m \Lambda_i(x_0) H_i(x_0, Dw_i) = 0, \text{ i.e. } H_i(x_0, Dw_i) = 0 \quad \text{for } i = 1, \dots, m.$$

Then, we obtain a linear system for w :

$$(2.10) \quad \frac{\partial w_i}{\partial t} + \sum_{j=1}^m d_{ij}(x_0) w_j(x_0, t) = 0, \text{ i.e. } w(x_0, t) = \exp(-tD(x_0))w(x_0, 0).$$

Since D is irreducible, from Lemma 2.1, 0 is a simple eigenvalue and all the nonzero eigenvalues have a positive real part. It follows that there exists a matrix A such that

$$\exp(-tD(x_0)) = A + O(e^{-rt}),$$

where $r > 0$ is the smallest real part of the nonzero eigenvalues. Therefore,

$$w(x_0, t) \xrightarrow{t \rightarrow +\infty} Aw(x_0, 0) + w_0 =: u_\infty(x_0).$$

It is now straightforward to see that $u_\infty(x_0)$ is the limit of $u(x_0, t)$. □

3. PROOF OF THEOREM 1.6

For this section, it is convenient to rewrite the stationary system under the form

$$(3.1) \quad F_i(x, Du_i) + \sum_{j=1}^m d_{ij} u_j = f_i + c_1 \quad x \in \mathbb{T}^N, \quad i = 1, \dots, m.$$

where $c = (c_1, \dots, c_1)$ is the ergodic constant. The existence of a solution to this system is given in Theorem 2.2.

Let $x_0 \in \mathcal{S}$, we recall that \mathcal{S} is defined by (1.40). We show that $c_1 = -\frac{\sum_{i=1}^m \Lambda_i f_i(x_0)}{\sum_{i=1}^m \Lambda_i}$ by constructing a constant solution for system (3.1) with respect to this constant.

Lemma 3.1. *The system*

$$(3.2) \quad F_i(x, Du_i) + \sum_{j=1}^m d_{ij} u_j = b_i - \frac{\sum_{i=1}^m \Lambda_i b_i}{\sum_{i=1}^m \Lambda_i}, \quad i = 1, \dots, m.$$

has a constant solution.

The proof is postponed.

Proof of Theorem 1.6. Step 1. Reduction to Lipschitz initial datas. Given $u_0 \in C(\mathbb{T}^N)^m$, set $S(t)u_0 = u(x, t)$ for $t \geq 0$ where u is the solution of (1.3) with initial datum u_0 . Then it is easy to see that $S(\cdot)$ generates a nonlinear, monotone, nonexpansive semigroup in $C(\mathbb{T}^N)^m$. Since $S(t)$ is nonexpansive, it is sufficient to show the result for $u_0 \in W^{1,\infty}(\mathbb{T}^N)$.

Step 2. Explicit value of the ergodic constant. Using the uniqueness of the ergodic constant and Lemma 3.1, we obtain the formula for the ergodic constant

$$(3.3) \quad c_1 = -\min_{x \in \mathbb{T}} \frac{\sum_{i=1}^m \Lambda_i f_i(x)}{\sum_{i=1}^m \Lambda_i} = -\frac{\sum_{i=1}^m \Lambda_i f_i(x_0)}{\sum_{i=1}^m \Lambda_i} \text{ for any } x_0 \in \mathcal{S},$$

where the constant vector $\Lambda = (\Lambda_1, \dots, \Lambda_m) > 0$ given by Lemma 2.1.

Step 3. Boundedness of $u + ct$ and use of stability results. Set $v := u + ct$. We apply Proposition 2.7 to deduce that v is bounded. Then, we can introduce the relaxed half-limits

$$(3.4) \quad \begin{aligned} \bar{v}(x) &= \limsup_{t \rightarrow +\infty}^* v(x, t) = \lim_{t \rightarrow +\infty} \sup\{v(y, s) : y \in B(x, 1/t), s \geq t\}, \\ \underline{v}(x) &= \liminf_{t \rightarrow +\infty}^* v(x, t) = \lim_{t \rightarrow +\infty} \inf\{v(y, s) : y \in B(x, 1/t), s \geq t\}, \end{aligned}$$

where the half-limits are taken componentwise.

By the stability result for viscosity solutions, \bar{v} and \underline{v} are respectively a sub and a supersolution of (3.1).

Step 4. Uniform convergence of the sequence $v(\cdot, t)$ on \mathcal{S} . We have that $v = u + ct$ is the solution of the system:

$$(3.5) \quad \begin{cases} \frac{\partial v_i}{\partial t} + F_i(x, Dv_i) + \sum_{j=1}^m d_{ij}v_j = f_i + c & (x, t) \in \mathbb{T}^N \times (0, +\infty), \\ v_i(x, 0) = u_{i0}(x) & x \in \mathbb{T}^N, \end{cases} \quad i = 1, \dots, m.$$

From Lemma 3.1, there exists a constant solution $w_0 = (w_{10}, \dots, w_{m0})$ for (2.10) with $b_i = f_i(x_0)$, $x_0 \in \mathcal{S}$ for $i = 1, \dots, m$. Set $w_i(x, t) = v_i(x, t) - w_{i0}$, then (w_1, \dots, w_m) solves

$$\frac{\partial w_i}{\partial t} + F_i(x, Dw_i) + \sum_{j=1}^m d_{ij}w_j = f_i(x) - f_i(x_0) \quad (x, t) \in \mathbb{T}^N \times (0, +\infty), \quad x_0 \in \mathcal{S}, \quad i = 1, \dots, m.$$

For this new system, the requirements of Lemma 2.8 are satisfied with K is chosen to be \mathcal{S} , hence we can apply it to have the convergence of $v_i(\cdot, t)$ as $t \rightarrow \infty$ for all $x \in \mathcal{S}$.

Step 5. Use of the comparison principle for (3.1) to conclude. The functions \bar{v} and \underline{v} are respectively a sub and a supersolution of (3.1) satisfying $\bar{v} \leq \underline{v}$ on \mathcal{S} . We have a comparison principle for the stationary system (3.1).

Theorem 3.2. *Assume that D is independent of x and satisfies (1.32). We assume moreover that (1.39) and (1.40) hold. Let $u \in \text{USC}(\mathbb{T}^N)$ and $v \in \text{LSC}(\mathbb{T}^N)$ be respectively a bounded subsolution and supersolution of (3.1) such that*

$$u_i(x) \leq v_i(x), \quad i \in \{1, \dots, m\}, \quad x \in \mathcal{S}.$$

Then

$$u \leq v \quad \text{in } \mathbb{T}^N.$$

The proof is postponed. Using this theorem, we infer that $\bar{v} \leq \underline{v}$ and hence, $\bar{v} = \underline{v} = v_\infty$. It proves the uniform convergence of $u + ct$ to u_∞ in \mathbb{T}^N and u_∞ solves (3.1). \square

Remark 3.3. When (1.10) holds, the right hand side of system (3.1) is nonnegative (that is, all its components are nonnegative). When (1.40) holds, the right hand side of the system (3.1) is $(f_1 + c_1, \dots, f_m + c_1)$ whose signs may not be nonnegative anymore, see Example 1.2.

The following example shows the difficulty when the coupling depends on x , even the f_i 's attain their minimum at the same point *but* with different values.

Example 3.4. Let $d \in C(\mathbb{T}^N)$ satisfying $\max_{\mathbb{T}^N} d > \min_{\mathbb{T}^N} d > 0$. We can show that there exists $c > -1$ and a bounded solution (v_1, v_2) to the system

$$(3.6) \quad \begin{cases} \frac{\partial v_1}{\partial t} + |Dv_1| + d(x)(v_1 - v_2) = c, \\ \frac{\partial v_2}{\partial t} + |Dv_2| + d(x)(v_2 - v_1) = 2 + c \end{cases} \quad x \in \mathbb{T}^N$$

(see Theorem 2.2). Note that, for (3.6), $\mathcal{F} = \emptyset$ but $f_1 \equiv 0$ and $f_2 \equiv 2$ achieve their minimum 0 and 2 respectively, at each point of \mathbb{T}^N so $\mathcal{S} = \mathbb{T}^N$. To follow the lines of the proof of convergence of [25, 6, 22], we need to identify a nonempty set $\tilde{\mathcal{S}}$ such that $v_i(x, t)$ converges as $t \rightarrow \infty$ for any $x \in \tilde{\mathcal{S}}$ and such that we have a comparison principle for (1.8) with prescribed values on $\tilde{\mathcal{S}}$. But, summing the equations in (3.6), we obtain

$$(3.7) \quad \frac{\partial(v_1 + v_2)}{\partial t} + |Dv_1| + |Dv_2| = 2 + 2c > 0,$$

which shows that it is hard to have such a set. We do not have the answer for this case.

We turn to the proof of the results used in the proof of Theorem 1.6.

Proof of Lemma 3.1. Since we construct constant solutions, we do not need to care about $F_i(x, Du_i) = F_i(x, 0) = 0$. Set $a := \frac{\sum_{i=1}^m \Lambda_i b_i}{\sum_{i=1}^m \Lambda_i}$. We want to find a solution of

$$\sum_{j=1}^m d_{ij} u_j = b_i - a, \quad i = 1, \dots, m.$$

We need to use

Lemma 3.5. *Assume that D is independent of x and satisfies (1.32). Then, the matrix obtaining after deleting the i th row and i th column is invertible.*

The proof is given at the end of this section.

From this lemma, we can find a unique solution (u_1, \dots, u_{m-1}) satisfying

$$\sum_{j=1}^{m-1} d_{ij} u_j = b_i - a, \quad i = 1, \dots, m-1.$$

Set $u_m = 0$, we have

$$(3.8) \quad \sum_{j=1}^m d_{ij} u_j = b_i - a, \quad i = 1, \dots, m-1.$$

We claim that (3.8) holds for $i = m$. Multiplying the i th equation in (3.8) by Λ_i and summing all equations for $i = 1, \dots, m-1$, we obtain

$$\sum_{i=1}^{m-1} \left(\sum_{j=1}^m \Lambda_i d_{ij} \right) u_j = \sum_{i=1}^{m-1} \Lambda_i b_i - \sum_{i=1}^{m-1} \Lambda_i a, \quad \text{i.e.,} \quad \sum_{j=1}^m -\Lambda_m d_{mj} u_j = \Lambda_m (a - b_m),$$

which is exactly what we need. □

Proof of Theorem 3.2. Proving directly this theorem seems hard because of the lackness of positivity in the right hand side of (3.1), see Remark 3.3. We instead try to bring it back to Theorem 3.2 with (1.10) holds instead of (1.40). To do so, the fact that the coupling matrix $D(x)$ independent of x plays a key role.

Using Lemma 3.1, there is a constant solution $u_0 = (u_{01}, \dots, u_{0m})$ of system (3.2) with $b_i = f_i(x_0)$. Writing system (3.1) as:

$$F_i(x, Du_i) + \sum_{j=1}^m d_{ij}u_j = (f_i - f_i(x_0)) + f_i(x_0) + c$$

and, setting $v_j = u_j - u_{0j}$, we obtain that $v = (v_1, \dots, v_m)$ solves:

$$F_i(x, Dv_i) + \sum_{j=1}^m d_{ij}v_j = f_i - f_i(x_0), \quad x \in \mathbb{T}^N, \quad i = 1, \dots, m.$$

Note that the g_i 's defined by $g_i(x) = f_i(x) - f_i(x_0)$ attain their minimum at the same point x_0 and their value at x_0 are equal to 0, i.e. \mathcal{F} satisfies (1.10) with $\mathcal{F} := \mathcal{S}$ and the f_i 's are replaced by the g_i 's. It follows that (1.10) holds, we then apply Theorem 3.2 with (1.10) holds to obtain the conclusion. \square

Proof of Lemma 3.5. Call $(\Lambda_i)_{i=1, \dots, m}$ the vector in Lemma 2.1. Call E the matrix after deleting the m th row and m th column from the matrix D , our goal is to prove that E is invertible. By contradiction, assume there exists $x = (x_1, \dots, x_{m-1}) \neq 0$ such that $Ex = 0$. It is straightforward to check that $y = (x, 0)$ satisfies $Dy = 0$, and from Lemma 2.1 again, we conclude that $x = 0$ which is a contradiction. Therefore, E is invertible. \square

4. PROOF OF THEOREM 1.7

Before giving the proof of Theorem 1.7, let us give typical examples

Example 4.1. A typical example of system (1.1) satisfying (1.41)-(1.42) is when

$$H_i(x, p) = f_i(x, p)|p|^{\alpha_i} + \sum_{1 \leq j \leq k} m_{ij}(|p + q_{ij}|^{a_{ij}} - |q_{ij}|^{a_{ij}}), \quad 1 \leq i \leq m,$$

where, for all i, j , $f_i \geq 0$, $\alpha_i, a_{ij} \geq 1$, $m_{ij} \geq 0$, $q_{ij} \in \mathbb{R}^N$ and there exists $1 \leq i_0 \leq m$ such that $f_{i_0}(x, p) \geq \eta > 0$ and $\alpha_{i_0} > 1$. In this case, H_{i_0} enters the framework of [8].

Let us point out that there are Hamiltonians satisfying (1.41)-(1.42) which are not covered by the results of [4], see Example 5.2 for details.

Theorem (1.7) improves a result of [22] where convergence for (1.43) with $\alpha = \beta = 2$ is proved. The proof of Theorem (1.7) is based on similar ideas as in [22]. The main change is that we can prove the proposition 4.3, which is the main estimate, for any strictly convex Hamiltonians after refining classical Jensen inequality.

Proof of Theorem 1.7.

Step 1. Boundedness of u . With assumption (1.41), we deduce that the ergodic constant is 0. In particular, the solution of (1.1) is bounded.

Step 2. The functions $U_{\max}(t) := \max_{x,i} u_i(x, t)$ and $U_{\min}(t) := \min_{x,i} u_i(x, t)$ are monotone. This result is proved in [22]. We make here a formal proof for U_{\max} . We fix t and assume,

without loss of generality, that $U_{\max}(t) = \max_{x,i} u_i(x, t) = u_1(x_0, t)$. Then

$$Du_1(x_0, t) = 0, \quad \sum_{j=1}^m d_{1j} u_j(x_0, t) \geq 0.$$

It follows from the first equation of (1.1) and (1.41) that

$$0 = \frac{\partial u_1}{\partial t}(x_0, t) + H_1(x_0, Du_1(x_0, t)) + \sum_{j=1}^m d_{1j} u_j(x_0, t) \geq U'_{\max}(t).$$

From Steps 1 and 2, we can set $L = \lim_{t \rightarrow \infty} U_{\max}(t)$ and $l = \lim_{t \rightarrow \infty} U_{\min}(t)$. If $L = l$, the proof is finished. We assume that $L > l$ and we will show a contradiction.

Step 3. Uniform convergence of a subsequence. From Lemma 2.8, there exists a sequence $t_n \rightarrow +\infty$ such that $(u(\cdot, t_n + \cdot))_n$ converges uniformly to some function $w \in W^{1,\infty}(\mathbb{T}^N \times [0, +\infty))$. By the stability of viscosity solutions, w is a solution of (1.1). By passing from u_i 's to w_i 's, we get new information. That is, we have

$$\max_{x,i} w_i(x, t) = L, \quad \min_{x,i} w_i(x, t) = l \quad \text{for all } t > 0.$$

Fix any $t > 0$. We have $\max_{x,i} w_i(x, t) = w_{i_0}(x_0, t) = L = \max_{x,s} w_{i_0}(x, s)$. Using 0 as a test function, from (1.1), we see that

$$\sum_{j=1}^m d_{i_0 j} w_j(x_0, t) \leq 0 \Rightarrow w_j(x_0, t) = w_{i_0}(x_0, t) \text{ for all } j,$$

where the last equality comes from the irreducibility of the coupling D . Indeed, we can use the following characterization of irreducibility ([6] and the references therein): D is irreducible if and only if for all $i, j \in \{1, \dots, m\}$, there exists $n \in \mathbb{N}$ and a sequence $i_0 = i, i_1, i_2, \dots, i_n = j$ such that $d_{i_{l-1} i_l} \neq 0$ for all $1 \leq l \leq n$ (in this case we say that there exists a chain between i and j). Then, for any $j \in \{1, \dots, m\}$, there exists $n \in \mathbb{N}$ and a sequence $i_1, i_2, \dots, i_n = j$ such that $d_{i_{l-1} i_l} \neq 0$ for all $1 \leq l \leq n$. From the above inequality and (1.2), we deduce that $w_{i_1}(x_0, t) = w_{i_0}(x_0, t)$ and repeat the same arguments, we obtain easily that $w_j(x_0, t) = \dots = w_{i_1}(x_0, t) = w_{i_0}(x_0, t)$.

This fact and a similar argument for the minimum leads to

$$(4.1) \quad \max_x w_i(x, t) = L, \quad \min_x w_i(x, t) = l \quad \text{for all } t > 0 \text{ and } i \in \{1, \dots, m\}.$$

Step 4. Strict convexity of one Hamiltonian implies the unboundedness of w_i 's. Using Lemma 2.1, we have

$$\frac{\partial}{\partial t} \Phi(x, t) + \sum_{i=1}^m \Lambda_i H_i(x, Dw_i) = 0, \quad \text{where } \Phi(x, t) = \sum_{i=1}^m \Lambda_i w_i(x, t).$$

Therefore

$$\frac{\partial}{\partial t} \Phi(x, t) + \rho(x) \sum_{i=1}^m \Lambda_i K_i(Dw_i) \leq 0.$$

It follows that

$$(4.2) \quad \frac{\partial}{\partial t} \int_{\mathbb{T}^N} \frac{\Phi(x, t)}{\rho(x)} dx + \int_{\mathbb{T}^N} \sum_{i=1}^m \Lambda_i K_i(Dw_i) dx \leq 0.$$

Next, we recall the Jensen inequality in the multidimensional case.

Proposition 4.2.

(i) Let ϕ be a convex function over an open bounded convex subset $U \subset \mathbb{R}^N$. For any integrable function $f : \mathbb{T}^N \rightarrow U$, we have

$$\int_{\mathbb{T}^N} \phi \circ f(x) dx \geq \phi \left(\int_{\mathbb{T}^N} f(x) dx \right).$$

(ii) If we assume moreover that ϕ is a strictly convex function, then, for any integrable function $f : \mathbb{T}^N \rightarrow U$ which is not constant a.e., we have

$$\int_{\mathbb{T}^N} \phi \circ f(x) dx > \phi \left(\int_{\mathbb{T}^N} f(x) dx \right).$$

The proof is given at the end of this section.

Part (i) of Proposition 4.2 yields $\int_{\mathbb{T}^N} K_i(Dw_i) dx \geq K_i[\int_{\mathbb{T}^N} Dw_i dx] \geq 0$ for all $i = 1, \dots, m$. And from (1.42), we can assume without loss of generality that K_1 is strictly convex. We have an important estimate for this strictly convex Hamiltonian

Proposition 4.3. Fix $L > l$ and a strictly convex function K satisfying $K(0) = 0$. Set

$$A = \{f \in W^{1,\infty}(\mathbb{T}^N) \text{ such that } \|f\|_{W^{1,\infty}(\mathbb{T}^N)} \leq C, \max_{\mathbb{T}^N} f = L, \min_{\mathbb{T}^N} f = l\}.$$

Then we have, for all $f \in A$,

$$\int_{\mathbb{T}^N} K(Df) dx \geq \beta > 0, \text{ where } \beta \text{ is independent of } f.$$

From Proposition 4.3, we derive

$$\int_{\mathbb{T}^N} \sum_{i=1}^m \Lambda_i K_i(Dw_i) dx \geq \int_{\mathbb{T}^N} \Lambda_1 K_1(Dw_1) dx \geq \delta > 0.$$

in which δ is independent of t thanks to (4.1). Using this for (4.2), we have

$$\frac{\partial}{\partial t} \int_{\mathbb{T}^N} \frac{\Phi(x, t)}{\rho(x)} dx \leq -\delta, \text{ for all } t.$$

This leads to $\frac{\Phi(x, t)}{\rho(x)} \rightarrow -\infty$ as t tends to infinity. This is a contradiction with the boundedness of w_i 's. It ends the proof of Theorem 1.7. \square

Now, we turn to the proof of the key estimate.

Proof of Proposition 4.3. Note that, by periodicity of f , $\int_{\mathbb{T}^N} Df = 0$. Therefore, by using Proposition 4.2 (i) and periodicity of f , we always have $\int_{\mathbb{T}^N} K(Df) dx \geq 0$.

We now assume by contradiction that such a β does not exist, therefore we can find a sequence $(f_n) \in A$ such that

$$\int_{\mathbb{T}^N} K(Df_n) dx < \frac{1}{n}.$$

Ascoli's theorem claims, by passing to a subsequence if necessary, the existence of $f_0 \in W^{1,\infty}(\mathbb{T}^N)$ so that

$$(4.3) \quad f_n \rightarrow f_0 \text{ in } C(\mathbb{T}^N).$$

In particular f_0 is not a constant because $\max_{\mathbb{T}^N} f_0 = L > \min_{\mathbb{T}^N} f_0 = l$.

Moreover, since $W^{1,2}(\mathbb{T}^N)$ is a reflexive Banach space, by passing to a subsequence if necessary

$$f_n \rightharpoonup g_0 \text{ in } W^{1,2}(\mathbb{T}^N).$$

Then, it follows that

$$(4.4) \quad f_n \rightharpoonup g_0 \text{ in } L^2(\mathbb{T}^N).$$

From (4.3) and (4.4), we obtain

$$g_0 = f_0 \text{ a.e.}$$

From a classical result in calculus of variations, see [7], we have

$$0 \leq \int_{\mathbb{T}^N} K(Df_0) \leq \liminf_n \int_{\mathbb{T}^N} K(Df_n) \leq 0.$$

Finally, we have

$$\int_{\mathbb{T}^N} K(Df_0) = 0.$$

We apply Proposition 4.2 (ii) to deduce that $Df_0(x) = 0$ a.e, and therefore $f_0(x) = C, \forall x \in \mathbb{T}^N$ by continuity. It leads to a contradiction. \square

Remark 4.4. Actually, the strict convexity of K was only used to deduce $Df_0(x) = 0$ a.e in last lines of the proof of Proposition 4.3. It follows that Proposition 4.3 can be easily established if K satisfies

$$K(p) \geq 0 \text{ for all } p \in \mathbb{R}^N, K \text{ is convex and } K(p) = 0 \Leftrightarrow p = 0.$$

Thus, Theorem 1.7 is still true if we replace the strict convexity with the above condition.

Proof of Proposition 4.2. (i) Set $t = \int_{\mathbb{T}^N} f(x)dx$, since ϕ is convex then its subdifferential $\partial\phi(t)$ is nonempty,

$$\phi(f(x)) \geq \phi(t) + \langle f(x) - t, p \rangle.$$

where $p \in \partial\phi(t)$. Summing after integrating both sides of the above inequality with respect to x and using the special value of t , we obtain the result.

For part (ii), since f is not constant a.e., the set $A = \{x \in U, f(x) \neq t\}$ has positive measure. Take $p \in \partial\phi(t)$, it is obvious that

$$\phi(f(x)) = \phi(t) + \langle f(x) - t, p \rangle \text{ if } x \notin A.$$

We prove later that

$$(4.5) \quad \phi(f(x)) > \phi(t) + \langle f(x) - t, p \rangle \text{ if } x \in A.$$

Integrating both sides of two above inequalities with respect to x and using the special value of t , we obtain the result.

To prove (4.5), we assume by contradiction there exists $x \in A$ such that

$$\phi(f(x)) = \phi(t) + \langle f(x) - t, p \rangle \text{ if } x \in A.$$

Therefore, using the strict convexity of ϕ and $f(x) \neq t$, we have

$$\phi\left(\frac{f(x)}{2} + \frac{t}{2}\right) < \frac{\phi(f(x))}{2} + \frac{\phi(t)}{2} = \phi(t) + \left\langle \frac{f(x) - t}{2}, p \right\rangle.$$

On the other hand, $p \in \partial\phi(t)$ and the convexity of ϕ imply

$$\phi\left(\frac{f(x)}{2} + \frac{t}{2}\right) \geq \phi(t) + \left\langle \frac{f(x) - t}{2}, p \right\rangle.$$

Two above inequalities give a contradiction. \square

5. PROOF OF THEOREM 1.9

Before proving Theorem 1.9, we introduce some assumptions to recall a result of [4] for single equations and its extension to the system (1.1) with $H_1 = H_2$ in [22].

$$(5.1) \quad \begin{cases} \text{Either } u \text{ is in } W^{1,\infty}(\mathbb{R}^N \times (0, \infty)) \text{ or there is a continuous function} \\ m : [0, \infty) \rightarrow [0, \infty) \text{ such that } m(0^+) = 0 \text{ and, for all } x, y, p \in \mathbb{R}^N, \\ |H(x, p) - H(y, p)| \leq m(|x - y|(1 + |p|)). \end{cases}$$

$$(5.2) \quad \begin{cases} \text{There exist } \eta > 0 \text{ and } \psi(\eta) > 0 \text{ such that: if } H(x, p + q) \geq \eta \text{ and} \\ H(x, q) \leq 0 \text{ for some } x \in A \subset \mathbb{R}^N, p, q \in \mathbb{R}^N \text{ then, for all } \mu \in (0, 1]: \\ \mu H(x, \mu^{-1}p + q) \geq H(x, p + q) + \psi(\eta)(1 - \mu), \end{cases}$$

$$(5.3) \quad \begin{cases} \text{There exists a, possibly empty, compact subset } K \text{ of } \mathbb{T}^N \text{ such that:} \\ (a) \ H(x, p) \geq 0 \text{ on } K \times \mathbb{R}^N, \\ (b) \text{ for all } \eta > 0, (5.2) \text{ holds with } A = K_\eta \text{ for all } \eta > 0, \\ \text{where } K_\eta := \{x \in \mathbb{R}^N : d(x, K) \geq \eta\}. \end{cases}$$

Theorem 5.1. ([4]) *Assume that $H \in C(\mathbb{T}^N \times \mathbb{R}^N)$ satisfies (5.1) and (5.3). Then, any solution $u \in \text{BUC}(\mathbb{T}^N \times (0, \infty))$ of (1.3) converges uniformly to a solution \bar{u} of (1.4).*

In [22], the authors proved the large time behavior of the solution of (1.1) with $m = 2$, $H_1 = H_2$ and H_i satisfies the conditions as in Theorem 5.1. The proof in [22] is based on the ideas used in the proof of Theorem 5.1. And therefore, it only works under the set of conditions on Hamiltonians of Theorem 5.1. But we observe that for this type of system, the convergence is actually inherited from the case of single equations.

Proof of Theorem 1.9. Step 1. Some estimates for single equations. We choose any sequence $(t_n)_n$ which tends to $+\infty$. For each $n \geq 0$, call Φ^n be the solution to the equation:

$$(5.4) \quad \begin{cases} \frac{\partial \Phi^n}{\partial t} + H(x, D\Phi^n) = 0 & (x, t) \in \mathbb{T}^N \times (0, +\infty), \\ \Phi^n(x, 0) = u_1(x, t_n) & x \in \mathbb{T}^N, \end{cases}$$

Note that (Φ^n, \dots, Φ^n) is a solution of (1.1) with initial conditions $(u_1(., t_n), \dots, u_1(., t_n))$. Applying the comparison principle for the system (1.1), we obtain

$$(5.5) \quad \sup_{i=1, \dots, m, x \in \mathbb{T}^N, t \geq 0} |u_i(x, t + t_n) - \Phi^n(x, t)| \leq \sup_{1 \leq i \leq m, x \in \mathbb{T}^N} |u_i(x, t_n) - u_1(x, t_n)|.$$

$$(5.6) \quad \sup_{x \in \mathbb{T}^N, t \geq 0} |\Phi^n(x, t) - \Phi^{n+1}(x, t)| \leq \sup_{x \in \mathbb{T}^N} |u_1(x, t_n) - u_1(x, t_{n+1})|.$$

Step 2. Use of the large time behavior result for single equations. The key idea is that for all $n \in \mathbb{N}$, the solutions of the scalar Hamilton-Jacobi equation (5.4) have the large time behavior by (1.44). It follows that

$$(5.7) \quad \Phi^n(., t) \rightarrow V^n(.) \text{ in } C(\mathbb{T}^N) \text{ as } t \rightarrow \infty \text{ for some } V^n \in \text{BUC}(\mathbb{T}^N)$$

and V^n is a solution of the equation (1.4).

Step 3. Derivation of the result for systems. From (5.6) and (5.7) we infer that $(V^n)_n$ is a Cauchy sequence in $\text{BUC}(\mathbb{T}^N)$ and therefore

$$(5.8) \quad V^n(\cdot) \rightarrow V(\cdot) \text{ in } C(\mathbb{T}^N) \text{ for some } V \in \text{BUC}(\mathbb{T}^N).$$

By the stability result, V is also a solution of (1.4). Using (5.7), we take \limsup^* with respect to t both sides of (5.5)

$$\overline{u}_i(x) \leq V^n(x) + \sup_{i=1, \dots, m, x \in \mathbb{T}^N} |u_i(x, t_n) - u_1(x, t_n)|, \text{ for all } i \text{ and } x \in \mathbb{T}^N.$$

From (5.8), Lemma 1.3, let n tend to infinity in the above inequality. We obtain

$$\overline{u}_i(x) \leq V(x) \text{ for all } i \text{ and } x \in \mathbb{T}^N.$$

Similarly, we get

$$\underline{u}_i(x) \geq V(x) \text{ for all } i \text{ and } x \in \mathbb{T}^N.$$

Hence, we conclude $u_i(\cdot, t) \rightarrow V(\cdot)$ in $C(\mathbb{T}^N)$ as t tends to infinity for all i . \square

Example 5.2. We show an example where the Hamiltonians satisfy the assumptions of Theorem 1.7 but not those of Theorems 1.1 and 5.1. It means that all the results of Section 4, even in the case of a single equation, are not covered by [4]. Take

$$H(x, p) = (2 + \sin(p))|p|^2 + |p + 1| - 1, \quad p \in \mathbb{R}.$$

We check that H does not satisfies (5.2) which is a key condition to prove Theorem 5.1. Since H is differentiable, (5.2) reduces to

$$(5.9) \quad \text{If } H(p + q) \geq \eta \text{ and } H(q) \leq 0 \text{ then } H_p(p + q)p - H(p + q) \geq \psi(\eta) > 0.$$

Choose $q = 0$, we then rewrite the above requirement as

$$H_p(p)p - H(p) = p[2p(2 + \sin(p)) + \cos(p)|p|^2] - (2 + \sin(p))|p|^2 + 1 - |p + 1| + p(1 - \frac{p + 1}{|p + 1|}).$$

Choose $p_n = 2n\pi + \pi$, $n \geq 0$, then we have

$$H_p(p_n)p_n - H(p_n) = 2p_n^2 - p_n^3 - p_n,$$

which tends to $-\infty$ as $n \rightarrow +\infty$. This violates (5.9). In the same way, we can check that H also does not satisfy the key hypothesis (1.7) of Theorem 1.1.

Proof of Lemma 1.3. This proof is a modified version of the one in [22] so that it can be adapted to general systems which is little more tricky.

Step 1. Some first estimates. Thanks to (1.34), we have

$$\delta = \min_{x \in \mathbb{T}^N, i, j=1, \dots, m, \mathcal{I} \subset \{1, \dots, m\}} -[\sum_{k \in \mathcal{I}} d_{ik}(x) + \sum_{k \in \mathcal{I}^c} d_{jk}(x)] > 0.$$

where \mathcal{I} contains j but not i .

Set $\Phi(t) = \max_{i \neq j, x \in \mathbb{T}^N} [u_i(x, t) - u_j(x, t)] \geq 0$ for each $t > 0$. Our purpose is to prove that Φ is a subsolution to the equation

$$(5.10) \quad \Phi'(t) + \delta\Phi(t) = 0.$$

Assume without loss of generality that $\Phi(t) = u_1(x_0, t) - u_2(x_0, t)$ and all functions are smooth to do a formal proof. It can be done rigorously by approximation techniques.

We have $\Phi'(t) = \frac{\partial u_1}{\partial t}(x_0, t) - \frac{\partial u_2}{\partial t}(x_0, t)$, $Du_1(x_0, t) = Du_2(x_0, t)$. Subtracting two first equations in (1.1), we have

$$\Phi'(t) + \sum_{j=1}^m d_{1j}(x_0)u_j(x_0, t) - \sum_{j=1}^m d_{2j}(x_0)u_j(x_0, t) = 0.$$

To obtain the conclusion, we only need to prove that

$$\sum_{j=1}^m [d_{1j}(x_0) - d_{2j}(x_0)]u_j(x_0, t) \geq \delta(u_1(x_0, t) - u_2(x_0, t))$$

or

$$(5.11) \quad (d_{11} - d_{21} - \delta)u_1 \geq (d_{22} - d_{12} - \delta)u_2 + \sum_{j=3}^m (d_{2j} - d_{1j})u_j.$$

At the point (x_0, t) , we have

$$(5.12) \quad u_1 \geq u_3, \dots, u_m \geq u_2, \quad d_{11} - d_{21} - \delta = d_{22} - d_{12} - \delta + \sum_{j=3}^m (d_{2j} - d_{1j})$$

but the signs of $d_{2j} - d_{1j}, j \geq 3$ are unknown.

Step 2. Separate the signs of $d_{2j} - d_{1j}, j \geq 3$. We call J^+ is the set consists of all $j \in \{3, \dots, m\}$ such that $d_{2j} - d_{1j} \geq 0$ and $J^- := \{3, \dots, m\} - J^+$. We rewrite (5.11) as

$$(d_{11} - d_{21} - \delta)u_1 - \sum_{j \in J^+} (d_{2j} - d_{1j})u_j \geq (d_{22} - d_{12} - \delta)u_2 + \sum_{j \in J^-} (d_{2j} - d_{1j})u_j$$

Actually, we can prove a stronger inequality

$$(d_{11} - d_{21} - \delta)u_1 - \sum_{j \in J^+} (d_{2j} - d_{1j})u_1 \geq (d_{22} - d_{12} - \delta)u_2 + \sum_{j \in J^-} (d_{2j} - d_{1j})u_2$$

It is clear by (5.12) that

$$d_{11} - d_{21} - \delta - \sum_{j \in J^+} (d_{2j} - d_{1j}) = d_{22} - d_{12} - \delta + \sum_{j \in J^-} (d_{2j} - d_{1j})$$

From this equality and $u_1 \geq u_2$, we only need to prove that

$$d_{11} - d_{21} - \delta - \sum_{j \in J^+} (d_{2j} - d_{1j}) \geq 0.$$

This is true by the definition of δ .

$$d_{11} - d_{21} - \sum_{j \in J^+} (d_{2j} - d_{1j}) = -[d_{12} + d_{21} + \sum_{j \in J^+} d_{2j} + \sum_{j \in J^-} d_{1j}] \geq \delta.$$

Since $\Phi(0)e^{-\delta t}$ is a supersolution of (5.10) with the initial value $\Phi(0)$, the comparison principle yields $0 \leq \Phi(t) \leq \Phi(0)e^{-\delta t}$ for all t . Therefore, $\Phi(t)$ converges to 0 as $t \rightarrow \infty$. \square

6. PROOF OF THE MAIN RESULT, THEOREM 1.4.

As we said in the introduction, to prove Theorem 1.4, we perform a change of function which leads to a new system (1.27). We start by giving a result for this auxiliary system, Theorem 6.1, which is used to prove Theorem 1.4. We introduce the assumption

$$(6.1) \quad w_i(x, t) \text{ converges as } t \rightarrow \infty \text{ for all } i = 1, \dots, m \text{ and } x \in \mathcal{K},$$

where \mathcal{K} is given in (1.28). It is convenient to state this assumption here. But, under the conditions of Theorem 1.4, this assumption is automatically fulfilled (see Step 1 of the proof).

For $\eta > 0$, we define

$$(6.2) \quad M_{\eta,i}(t) = \sup_{x \in \mathbb{T}^N, s \geq t} [w_i(x, t) - w_i(x, s) - 2\eta(s - t)].$$

We state the key estimate on this function

Theorem 6.1. *Consider system (1.27) where the coupling D is independent of x and satisfies (1.2) and (1.33). Assume that $F_i \in C(\mathbb{T}^N \times \mathbb{R} \times \mathbb{R}^N)$, $w_i \in W^{1,\infty}(\mathbb{T}^N \times (0, +\infty))$ and that (1.28), (6.1) hold. Taking a sequence $t_n \rightarrow +\infty$ such that $(w(\cdot, t_n + \cdot))_n$ converges uniformly to some function $v \in W^{1,\infty}(\mathbb{T}^N \times [0, \infty))^m$. Define*

$$(6.3) \quad P_{\eta,i}(t) = \sup_{x \in \mathbb{T}^N, s \geq t} [v_i(x, t) - v_i(x, s) - 2\eta(s - t)].$$

Then

$$(6.4) \quad P_{\eta,i}(t) = 0 \text{ for any } i = 1, \dots, m, t \geq 0 \text{ and } \eta > 0.$$

We first give the proof of Theorem 1.4 under the set of assumptions (1.25) using Theorem 6.1. Then we prove Theorem 6.1. Notice that we skip the proof of Theorem 1.4 under (1.26) for shortness since it can be deduced from the arguments under (1.25).

Proof of Theorem 1.4. Step 1. Since the solution u of (1.1) is bounded, we can assume, by adding a big enough positive constant on the initial conditions if needed, that

$$M \geq u_i(x, t) \geq 1, \quad x \in \mathbb{T}^N, \quad t > 0, \quad i = 1, \dots, m.$$

Set

$$(6.5) \quad \exp(w_i(x, t)) = u_i(x, t) \text{ for all } i = 1, \dots, m \text{ and } (x, t) \in \mathbb{T}^N \times (0, \infty).$$

Then $w_i \in W^{1,\infty}(\mathbb{T}^N \times (0, +\infty))$ for $i = 1, \dots, m$, solves (1.27) with

$$F_i(x, w, p) = \exp(-w)H_i(x, \exp(w)p).$$

We can check that F_i satisfies (1.28) with $\mathcal{K} := K$ where K is given in (1.25). Moreover, Lemma 2.8 gives the convergence of $u_i(x, t)$, as $t \rightarrow \infty$ for all $x \in K$. Thus (6.1) is automatically satisfied from the way we define w_i .

Step 2. We apply Theorem 6.1 to (1.27). By the stability result, the function v defined as in the statement of Theorem 6.1 is a viscosity solution of (1.27). By (6.4),

$$v_i(x, t) - v_i(x, s) - 2\eta(s - t) \leq 0, \text{ for all } i, s \geq t \text{ and } x \in \mathbb{T}^N.$$

Letting η tend to 0, we obtain

$$v_i(x, t) - v_i(x, s) \leq 0, \text{ for all } i, x \in \mathbb{T}^N, s \geq t \geq 0.$$

Step 3. The uniform convergence of $(w(\cdot, t_n + \cdot))_n$ to $v \in W^{1,\infty}(\mathbb{T}^N \times [0, +\infty))$ yields

$$-o_n(1) + v_i(x, t) \leq w_i(x, t + t_n) \leq o_n(1) + v_i(x, t) \text{ in } \mathbb{T}^N \times (0, \infty), \text{ for all } i.$$

Since v_i is nondecreasing in t , $v_i(x, t) \rightarrow v_{i\infty}(x)$ uniformly in x as t tends to infinity. Taking \limsup^* and \liminf_* with respect to t both sides of the above estimate, we obtain

$$-o_n(1) + v_{i\infty}(x) \leq \liminf_{t \rightarrow +\infty}^* w_i(x, t) \leq \limsup_{t \rightarrow +\infty}^* w_i(x, t) \leq o_n(1) + v_{i\infty}(x) \quad x \in \mathbb{T}^N, \text{ for all } i.$$

Letting n tend to infinity, we derive

$$\liminf_{t \rightarrow +\infty}^* w_i(x, t) = \limsup_{t \rightarrow +\infty}^* w_i(x, t) = v_{i\infty}(x), \quad x \in \mathbb{T}^N, \text{ for all } i,$$

which yields the uniform convergence of $w_i(\cdot, t)$ to $v_{i\infty}$ in \mathbb{T}^N as t tends to infinity.

Step 4. By stability, $v_{i\infty}$ is a solution of (1.27). Therefore, $u_i(\cdot, t) = \exp(w_i(\cdot, t))$ converges uniformly, as t tends to infinity, to $\exp(v_{i\infty})$ which is a solution of (1.8) with $(c_1, \dots, c_m) = 0$ under our assumption. It ends the proof of Theorem 1.4. \square

Now, we turn to the proof of Theorem 6.1.

Proof of Theorem 6.1. Step 1. We notice that $M_{\eta,i}$ introduced in (6.2) is nonnegative, bounded uniformly continuous thanks to the fact that $w_i \in W^{1,\infty}(\mathbb{T}^N \times (0, +\infty))$ for $i = 1, \dots, m$. We need the following result, the proof of which is postponed

Lemma 6.2. *Under the assumptions of Theorem 6.1.*

(i) $M_{\eta,i}(t)$ defined by (6.2) converges to the same limit for all i as t tends to infinity. An easy consequence of this fact is that $P_{\eta,i}(t) = c(\eta)$ where $c(\eta)$ depends only on η .

(ii) The $P_{\eta,i}$'s attain their maximum at the same point (x_t, s_t) for all i and

$$(6.6) \quad \begin{aligned} P_{\eta,i}(t) &= v_i(x_t, t) - v_i(x_t, s_t) - 2\eta(s_t - t) \\ &= P_{\eta,j}(t) = v_j(x_t, t) - v_j(x_t, s_t) - 2\eta(s_t - t) \text{ for all } i, j = 1, \dots, m \text{ and } t > 0. \end{aligned}$$

Remark 6.3. Let us emphasize that Lemma 6.2 (ii) is the key idea to prove Theorem 1.4. It is the rigorous expression of the idea which was explained in the Introduction (see (1.29)) to overcome the difficulty when passing from scalar equations to systems.

To prove Lemma 6.2, we only need (1.28)(i) and (1.28)(ii) is not used in that proof. The important condition (1.28)(ii) only plays a role in the proof of Theorem 6.1.

Step 2. From (6.1) and the definition of v_i , we have

$$(6.7) \quad v_i(x, t) \text{ is independent of } t \text{ for all } i = 1, \dots, m \text{ and } x \in \mathcal{K}.$$

From Lemma 6.2, for any fixed $\tau > 0$, there exists x_τ, s_τ satisfying (6.6). We choose $i \in \{1, \dots, m\}$ such that

$$(6.8) \quad v_i(x_\tau, s_\tau) = \min_{j=1, \dots, m} v_j(x_\tau, s_\tau).$$

Let $\Phi \in C^1((0, \infty))$ such that τ is a strict maximum point of $P_{\eta,i} - \Phi$ in $[\tau - \delta, \tau + \delta]$ for some $\delta > 0$. Since $P_{\eta,i}(\cdot)$ is constant, we have $\Phi'(\tau) = 0$. To prove the theorem, we assume by contradiction that $P_{\eta,i}(\tau) = c(\eta) > 0$. This fact and (6.7) imply that

$$(6.9) \quad d(x_\tau, \mathcal{K}) \geq \beta_\eta > 0, \text{ where } x_\tau \text{ satisfies (6.6) with } t = \tau.$$

and \mathcal{K} is defined in (1.28).

Step 3. Consider, $x, y \in \mathbb{T}^N$, $t \in [\tau - \delta, \tau + \delta]$, $s \geq t$ and the test function:

$$\Psi^{i,\epsilon}(x, y, t, s) = v_i(x, t) - v_i(y, s) - 2\eta(s - t) - |x - x_\tau|^2 - |s - s_\tau|^2 - \frac{|x - y|^2}{2\epsilon^2} - \Phi(t).$$

The function $\Psi^{i,\epsilon}$ achieves its maximum over $\mathbb{T}^N \times \mathbb{T}^N \times \{(t, s)/t \leq s, t \in [\tau - \delta, \tau + \delta]\}$ at $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$ because v_i is bounded continuous. We obtain some classical estimates when $\epsilon \rightarrow 0$:

$$(6.10) \quad \begin{cases} \Psi^{i,\epsilon}(\bar{x}, \bar{y}, \bar{t}, \bar{s}) \rightarrow P_{\eta,i}(\tau) - \Phi(\tau), \quad \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} \rightarrow 0, \\ (\bar{x}, \bar{s}, \bar{t}) \rightarrow (x_\tau, s_\tau, \tau) \text{ since } \tau \text{ is a strict maximum point of } P_{\eta,i} - \Phi, \\ v_i(\bar{x}, \bar{t}) - v_i(\bar{y}, \bar{s}) \geq P_{\eta,i}(\bar{t}), \\ \bar{s} > \bar{t} \text{ since } P_{\eta,i}(\tau) = c(\eta) > 0. \end{cases}$$

Step 4. Since v is the solution of (1.27), we can write the viscosity inequalities

$$\begin{cases} \Phi'(\bar{t}) - 2\eta + F_i(\bar{x}, v_i(\bar{x}, \bar{t}), p) + \sum_{j=1}^m d_{ij} \exp(v_j - v_i)(\bar{x}, \bar{t}) \leq 0, \\ -2\eta + F_i(\bar{y}, v_i(\bar{y}, \bar{s}), p) + \sum_{j=1}^m d_{ij} \exp(v_j - v_i)(\bar{y}, \bar{s}) \geq 0, \end{cases}$$

where $p = \frac{\bar{x} - \bar{y}}{\epsilon^2} + 2(\bar{x} - x_\tau)$. Note that v_i is bounded Lipschitz continuous, then $|F_i(\bar{x}, v_i(\bar{y}, \bar{s}), p) - F_i(\bar{y}, v_i(\bar{y}, \bar{s}), p)| \leq m(|x - y|) \leq O(\epsilon)$, thanks to the uniform continuity of F_i over compact subsets. It follows from (6.10),

$$(6.11) \quad \begin{cases} \Phi'(\bar{t}) - 2\eta + F_i(\bar{x}, v_i(\bar{x}, \bar{t}), p) + \sum_{j=1}^m d_{ij} \exp(v_j - v_i)(\bar{x}, \bar{t}) \leq 0, \\ -2\eta + F_i(\bar{x}, v_i(\bar{y}, \bar{s}), p) + \sum_{j=1}^m d_{ij} \exp(v_j - v_i)(\bar{x}, \bar{s}) + O(\epsilon) \geq 0. \end{cases}$$

Step 5. From (6.9), we have

$$(6.12) \quad d(\bar{x}, \mathcal{K}) \geq \frac{\beta_\eta}{2} \text{ for } \epsilon \text{ small enough.}$$

By (1.2), (6.8) and the fact that $\bar{s} \rightarrow s_\tau, \bar{x} \rightarrow x_\tau$ as $\epsilon \rightarrow 0$, we have

$$\sum_{j=1}^m d_{ij} \exp(v_j - v_i)(\bar{x}, \bar{s}) \leq O(\epsilon).$$

This gives

$$F_i(\bar{x}, v_i(\bar{y}, \bar{s}), p) \geq \eta > 0 \text{ for } \epsilon \text{ small enough.}$$

Thanks to (1.28) (ii) and the Lipschitz continuity of F_i with respect to w , we infer

$$\begin{aligned} F_i(\bar{x}, u, p) &\geq \eta > 0, \quad \text{for all } u \geq v_i(\bar{y}, \bar{s}), \\ (F_i)_w(\bar{x}, u, p) &\geq \Psi(\eta) > 0, \quad \text{for almost all } u \geq v_i(\bar{y}, \bar{s}). \end{aligned}$$

Since $v_i(\bar{x}, \bar{t}) \geq v_i(\bar{y}, \bar{s})$, we obtain

$$F_i(\bar{x}, v_i(\bar{x}, \bar{t}), p) - F_i(\bar{x}, v_i(\bar{y}, \bar{s}), p) \geq \Psi(\eta)(v_i(\bar{x}, \bar{t}) - v_i(\bar{y}, \bar{s})) \geq \Psi(\eta)P_{\eta,i}(\bar{t}).$$

Step 6. We compute

$$\begin{aligned} &\sum_{j=1}^m d_{ij} \exp(v_j - v_i)(\bar{x}, \bar{t}) - \sum_{j=1}^m d_{ij} \exp(v_j - v_i)(\bar{x}, \bar{s}) \\ &\geq \sum_{j \neq i} -d_{ij} \exp(v_j - v_i)(x_\tau, s_\tau) \{1 - \exp[P_{\eta,j}(\tau) - P_{\eta,i}(\tau)]\} + O(\epsilon) \\ &\geq O(\epsilon), \text{ since } P_{\eta,i}(\cdot) = P_{\eta,j}(\cdot) = c(\eta). \end{aligned}$$

Therefore, by subtracting both sides in (6.11), we get

$$\Phi'(\bar{t}) + \Psi(\eta)P_{\eta,i}(\bar{t}) + O(\epsilon) \leq 0.$$

Using (6.10) and the fact that $\Phi'(\tau) = 0$. Letting ϵ tend to 0, we obtain

$$\Psi(\eta)c(\eta) \leq 0, \text{ i.e. } c(\eta) \leq 0 \text{ which is a contradiction.}$$

□

Proof of Lemma 6.2. Proof of part (i). We need

Lemma 6.4. *Under the assumptions of Theorem 6.1, $N_{\eta,i} := \exp(M_{\eta,i})$ are subsolution of*

$$(6.13) \quad \min\{N'_{\eta,i} + \sum_{j \neq i}^m d_{ij}\rho_{ij}(t)[N_{\eta,j}(t) - N_{\eta,i}(t)], N_{\eta,i} - 1\} \leq 0, \quad i = 1, \dots, m,$$

where ρ_{ij} are functions defined by

$$\rho_{ij}(t) = \begin{cases} m_1 & \text{if } N_i(t) \geq N_j(t), \\ m_2 & \text{if } N_i(t) < N_j(t), \end{cases}$$

for all $i \neq j = 1, \dots, m$ and $t > 0$, and $m_1 \leq m_2$ are positive constants.

Set $N_{\eta,i} = \exp(M_{\eta,i}) \geq 1$. We show that (6.13) will lead to

$$N'_{\eta,i}(t) + \sum_{j \neq i}^m d_{ij}\rho_{ij}(t)[N_{\eta,j}(t) - N_{\eta,i}(t)] \leq 0, \quad i = 1, \dots, m,$$

Otherwise, we can assume that for $i = 1$, there is a function $\Phi \in C^1(\mathbb{R})$ such that $N_{\eta,1} - \Phi(t)$ attains its maximum at some $t_0 > 0$ and,

$$\Phi'(t_0) + \sum_{j=2}^m d_{1j}\rho_{1j}(t_0)[N_{\eta,j}(t_0) - N_{\eta,1}(t_0)] > 0.$$

From (6.13), we have $N_{\eta,1}(t_0) = 1$ and hence t_0 is a minimum point of Φ since $N_{\eta,1} \geq 1$. Finally, we get

$$\sum_{j=2}^m d_{1j}\rho_{1j}(t_0)[N_{\eta,j}(t_0) - 1] > 0,$$

this is a contradiction since $d_{1j} \leq 0$ for $j \geq 2$. We then use the following lemma to conclude

Lemma 6.5. *Let N_j be bounded, positive continuous functions satisfying*

$$(6.14) \quad N'_i + \sum_{j \neq i, j=1}^m d_{ij}\rho_{ij}(t)[N_j(t) - N_i(t)] \leq 0, \quad i = 1, \dots, m,$$

where $D = (d_{ij})_{1 \leq i, j \leq m}$ satisfies (1.2) and (1.33), ρ_{ij} are functions defined by

$$\rho_{ij}(t) = \begin{cases} 1 & \text{if } N_i(t) \geq N_j(t), \\ 2 & \text{if } N_i(t) < N_j(t), \end{cases}$$

for all $i \neq j = 1, \dots, m$. Then $N_i(t)$'s converge to the same limit as t tends to infinity.

The proof of Lemma 6.5 is quite technical since we cannot deal directly with the discontinuity of the ρ_{ij} 's. The main idea of the proof is that we reorder the functions N_i 's into the biggest function, the second biggest function, etc. and the smallest function (see Proof of Lemma 6.5 for details). Surprisingly, the new functions satisfy a nicer system where the discontinuous functions ρ_{ij} are replaced by constants. For this new system, we can prove

they converge to the same limit and, as a result, the old functions which are bounded by the biggest function and the smallest function must converge to the same limit. The proof of this Lemma will be given later.

Finally, since $w_i \rightarrow v_i$ uniformly as $t \rightarrow \infty$, we easily obtain that $P_{\eta,i}(t) = c(\eta)$ for all i and t , where $c(\eta)$ depends only on η . It completes the proof of part (i) of Lemma 6.2.

Proof of part (ii) of Lemma 6.2.

Step 1. Fix $\tau > 0$. If $P_{\eta,j}(\tau) = 0$, then we finish the proof since we can choose $s_\tau = \tau$ and any x_τ in (6.6) to fulfill the requirement. We therefore assume that $P_{\eta,1}(\tau) > 0$ and let $\Phi \in C^1((0, \infty))$ such that τ is a strict maximum point of $P_{\eta,1} - \Phi$ in $[\tau - \delta, \tau + \delta]$ for some $\delta > 0$. Since $P_{\eta,1}(\cdot)$ is a constant, $\Phi'(\tau) = 0$. Assume $P_{\eta,1}(\tau)$ attains its maximum at x_τ, s_τ .

Step 2. Consider, $x, y \in \mathbb{T}^N, t \in [\tau - \delta, \tau + \delta]$ and $s \geq t$ the test function:

$$\Psi^{1,\epsilon}(x, y, t, s) = v_1(x, t) - v_1(y, s) - 2\eta(s - t) - |x - x_\tau|^2 - |s - s_\tau|^2 - \frac{|x - y|^2}{2\epsilon^2} - \Phi(t).$$

Assume that $\Psi^{1,\epsilon}$ achieves its maximum over $\mathbb{T}^N \times \mathbb{T}^N \times \{(t, s)/t \leq s, t \in [\tau - \delta, \tau + \delta]\}$ at $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$. We obtain some classical estimates when $\epsilon \rightarrow 0$,

$$(6.15) \quad \begin{cases} \Psi^{1,\epsilon}(\bar{x}, \bar{y}, \bar{t}, \bar{s}) \rightarrow P_{\eta,1}(\tau) - \Phi(\tau), & \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} \rightarrow 0, \\ (\bar{x}, \bar{s}, \bar{t}) \rightarrow (x_\tau, s_\tau, \tau) \text{ since } \tau \text{ is a strict maximum point of } P_{\eta,1} - \Phi, \\ v_1(\bar{x}, \bar{t}) - v_1(\bar{y}, \bar{s}) \geq P_{\eta,1}(\bar{t}), \text{ and } \bar{s} > \bar{t} \text{ since } P_{\eta,1}(\tau) > 0. \end{cases}$$

Step 3. Since v is the solution of (1.27), we have

$$(6.16) \quad \begin{cases} \Phi'(\bar{t}) - 2\eta + F_1(\bar{x}, v_1(\bar{x}, \bar{t}), p) + \sum_{j=1}^m d_{1j} \exp(v_j - v_1)(\bar{x}, \bar{t}) \leq 0, \\ -2\eta + F_1(\bar{x}, v_1(\bar{y}, \bar{s}), p) + \sum_{j=1}^m d_{1j} \exp(v_j - v_1)(\bar{x}, \bar{s}) + O(\epsilon) \geq 0. \end{cases}$$

where $p = \frac{\bar{x} - \bar{y}}{\epsilon^2}$.

Step 4. Set $Q_{\eta,j}(\tau) := v_j(x_\tau, \tau) - v_j(x_\tau, s_\tau) - 2\eta(s_\tau - \tau)$ for $j \neq 1$, we have

$$\begin{aligned} & \sum_{j=1}^m d_{1j} \exp(v_j - v_1)(\bar{x}, \bar{t}) - \sum_{j=1}^m d_{1j} \exp(v_j - v_1)(\bar{x}, \bar{s}) \\ &= \sum_{j=2}^m -d_{1j} \exp(v_j - v_1)(x_\tau, s_\tau) \{1 - \exp[Q_{\eta,j}(\tau) - P_{\eta,1}(\tau)]\} + O(\epsilon). \end{aligned}$$

Using (1.28)(i) and the fact that $v_1(\bar{x}, \bar{t}) - v_1(\bar{y}, \bar{s}) \geq P_{\eta,1}(\bar{t}) \geq 0$, we have

$$F_1(\bar{x}, v_1(\bar{x}, \bar{t}), p) - F_1(\bar{x}, v_1(\bar{y}, \bar{s}), p) \geq 0.$$

Therefore, by subtracting both sides in (6.16), we get

$$\Phi'(\bar{t}) + \sum_{j=2}^m -d_{1j} \exp(v_j - v_1)(x_\tau, s_\tau) \{1 - \exp[Q_{\eta,j}(\tau) - P_{\eta,1}(\tau)]\} + O(\epsilon) \leq 0.$$

From (6.15), by letting $\epsilon \rightarrow 0$, we obtain

$$\sum_{j=2}^m -d_{1j} \exp(v_j - v_1)(x_\tau, s_\tau) \{1 - \exp[Q_{\eta,j}(\tau) - P_{\eta,1}(\tau)]\} \leq 0.$$

Since $-d_{1j} \geq 0$ for $j = 2, \dots, m$ and $Q_{\eta,j}(\tau) \leq P_{\eta,1}(\tau)$, it follows from the above inequality that $Q_{\eta,j}(\tau) = P_{\eta,1}(\tau)$, i.e., $P_{\eta,i}$'s attain their maximum at the same point (x_τ, s_τ) . \square

Proof of Lemma 6.4. We fix $i \in \{1, \dots, m\}$.

Step 1. Let $\Phi \in C^1((0, \infty))$ and assume that τ is a strict maximum point of $M_{\eta,i} - \Phi$ over $[\tau - \delta, \tau + \delta]$ for some $\delta > 0$. Everything will be done if $M_{\eta,i}(\tau) = 0$. We therefore assume that $M_{\eta,i}(\tau) > 0$.

Step 2. Consider, $x, y \in \mathbb{T}^N, t \in [\tau - \delta, \tau + \delta]$ and $s \geq t$ the test function:

$$\Psi^{i,\epsilon}(x, y, t, s) = w_i(x, t) - w_i(y, s) - 2\eta(s - t) - \frac{|x - y|^2}{2\epsilon^2} - \Phi(t).$$

Assume that $\Psi^{i,\epsilon}$ achieves its maximum over $\mathbb{T}^N \times \mathbb{T}^N \times \{(t, s)/t \leq s, t \in [\tau - \delta, \tau + \delta]\}$ at $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$. We obtain some classical estimates when $\epsilon \rightarrow 0$,

$$(6.17) \quad \begin{cases} \Psi^{i,\epsilon}(\bar{x}, \bar{y}, \bar{t}, \bar{s}) \rightarrow M_{\eta,i}(\tau) - \Phi(\tau), \quad \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} \rightarrow 0, \\ \bar{t} \rightarrow \tau \text{ since } \tau \text{ is a strict maximum point of } M_{\eta,i} - \Phi \text{ in } [\tau - \delta, \tau + \delta], \\ w_i(\bar{x}, \bar{t}) - w_i(\bar{y}, \bar{s}) \geq M_{\eta,i}(\bar{t}), \quad \bar{s} > \bar{t} \text{ since } M_{\eta,i}(\tau) > 0. \end{cases}$$

Step 3. Since w is the solution of (1.27), we have

$$\begin{cases} \Phi'(\bar{t}) - 2\eta + F_i(\bar{x}, w_i(\bar{x}, \bar{t}), p) + \sum_{j=1}^m d_{ij} \exp(w_j - w_i)(\bar{x}, \bar{t}) \leq 0, \\ -2\eta + F_i(\bar{x}, w_i(\bar{y}, \bar{s}), p) + \sum_{j=1}^m d_{ij} \exp(w_j - w_i)(\bar{x}, \bar{s}) + O(\epsilon) \geq 0. \end{cases}$$

where $p = \frac{\bar{x} - \bar{y}}{\epsilon^2}$.

Step 4. Using (1.28) and the fact that $w_i(\bar{x}, \bar{t}) \geq w_i(\bar{y}, \bar{s}) \geq M_{\eta,i}(\bar{t}) \geq 0$, we get

$$F_i(\bar{x}, w_i(\bar{x}, \bar{t}), p) - F_i(\bar{x}, w_i(\bar{y}, \bar{s}), p) \geq 0.$$

We compute

$$\begin{aligned} & \sum_{j=1}^m d_{ij} \exp(w_j - w_i)(\bar{x}, \bar{t}) - \sum_{j=1}^m d_{ij} \exp(w_j - w_i)(\bar{x}, \bar{s}) \\ & \geq \sum_{j \neq i} -d_{ij} \exp(w_j - w_i)(\bar{x}, \bar{s}) \{1 - \exp[M_{\eta,j}(\tau) - M_{\eta,i}(\tau)]\} + O(\epsilon) \\ & \geq \sum_{j \neq i} -d_{ij} \rho_{ij}(\tau) \{1 - \exp[M_{\eta,j}(\tau) - M_{\eta,i}(\tau)]\} + O(\epsilon). \end{aligned}$$

where we define for $i \neq j$

$$\rho_{ij}(t) := \begin{cases} m_1 & \text{if } 1 - \exp[M_{\eta,j}(t) - M_{\eta,i}(t)] \geq 0, \\ m_2 & \text{if } 1 - \exp[M_{\eta,j}(t) - M_{\eta,i}(t)] \leq 0. \end{cases}$$

with

$$\begin{aligned} m_1 &= \inf_{x \in \mathbb{T}^N, s > 0, 1 \leq i, j \leq m} \exp(w_j - w_i)(x, s) > 0, \\ m_2 &= \sup_{x \in \mathbb{T}^N, s > 0, 1 \leq i, j \leq m} \exp(w_j - w_i)(x, s) < \infty \end{aligned}$$

which are well-defined thanks to the boundedness of w_i .

Therefore, by subtracting both sides in step 4 we get:

$$\Phi'(\bar{t}) + \sum_{j \neq i} -d_{ij} \rho_{ij}(\tau) \{1 - \exp[M_{\eta,j}(\tau) - M_{\eta,i}(\tau)]\} + O(\epsilon) \leq 0.$$

Using (6.17) and letting ϵ tend to 0, we obtain

$$\Phi'(\bar{t}) + \sum_{j=1, j \neq i}^m -d_{ij}\rho_{ij}(\tau)\{1 - \exp[M_{\eta,j}(\tau) - M_{\eta,i}(\tau)]\} \leq 0,$$

which is exactly the viscosity inequality for (6.13) knowing $M_{\eta,i}(\tau) = \exp[M_{\eta,j}(\tau)] > 1$. \square

Proof of Lemma 6.5. Step 1. In general, the discontinuity of ρ_{ij} causes a lot of problems but since we are in a nicer situation, we can replace ρ_{ij} by constants with the following technique. For each t , we define

$$\begin{aligned} (6.18) \quad R_1(t) &= \max_{i \in \{1, \dots, m\}} N_i(t) := N_{i_1}(t), \\ R_2(t) &= \max_{i \in \{1, \dots, m\} - \{i_1\}} N_i(t) := N_{i_2}(t), \\ R_k(t) &= \max_{i \in \{1, \dots, m\} - \{i_1, \dots, i_{k-1}\}} N_i(t) := N_{i_k}(t), \quad k = 2, \dots, m. \end{aligned}$$

Strictly speaking, we reorder the functions N_i 's such that the new functions satisfy a nicer system where the discontinuous functions ρ_{ij} are replaced by constants.

Step 2. Let us assume for a while that R_i satisfies

$$(6.19) \quad R'_i(t) + \sum_{j=1}^m d'_{ij}R_j(t) \leq 0, \quad i = 1, \dots, m,$$

where $(d'_{ij})_{1 \leq i, j \leq m}$ satisfies (1.2) and (1.33). This fact will be proven at the end of this proof.

Step 3. Call Λ_i be the vector from Lemma 2.1, where the coupling now is $(d'_{ij})_{1 \leq i, j \leq m}$. We have

Lemma 6.6. ([6, Lemma 5.5]) $\sum_{j=1}^m \Lambda_j R_j(t)$ is nonincreasing and converges as $t \rightarrow +\infty$.

The idea of the proof of Lemma 6.6 was given in the proof of Lemma 2.8. From (1.33) for (d'_{ij}) , there exists $i \in \{1, \dots, m\}$ and $\alpha > 0$ such that

$$(6.20) \quad d'_{ij} + \alpha \Lambda_j < 0 \quad j = 1, \dots, m, \quad j \neq i.$$

From (6.19), we obtain that R_i is a subsolution of

$$R'_i(t) + (d'_{ii} + \alpha \Lambda_i)R_i \leq \alpha \Lambda_i R_i + \sum_{j \neq i} (-d'_{ij})R_j, \quad t \in (0, +\infty).$$

From the stability result ([1, 2, 9]), $\bar{R}_i = \limsup_{t \rightarrow +\infty}^* R_i(t)$ is a subsolution of

$$(6.21) \quad (d'_{ii} + \alpha \Lambda_i)\bar{R}_i \leq \limsup_{t \rightarrow +\infty}^* \left\{ \alpha \Lambda_i R_i + \sum_{j \neq i} (-d'_{ij})R_j \right\}.$$

But,

$$\begin{aligned}
& \limsup_{t \rightarrow +\infty}^* \{ \alpha \Lambda_i R_i + \sum_{j \neq i} (-d'_{ij}) R_j \} \\
& \leq \alpha \limsup_{t \rightarrow +\infty}^* \{ \sum_{j=1}^m \Lambda_j R_j \} + \sum_{j \neq i} \limsup_{t \rightarrow +\infty}^* \{ (-d'_{ij} - \alpha \Lambda_j) R_j \} \\
& \leq \alpha \limsup_{t \rightarrow +\infty}^* \{ \sum_{j=1}^m \Lambda_j R_j \} + \sum_{j \neq i} (-d'_{ij} - \alpha \Lambda_j) \bar{R}_j,
\end{aligned}$$

since (6.20) holds. The previous inequality and (6.21) imply

$$\alpha \sum_{j=1}^m \Lambda_j \bar{R}_j = (d'_{ii} + \alpha \Lambda_i) \bar{R}_i + \sum_{j \neq i} (d'_{ij} + \alpha \Lambda_j) \bar{R}_j \leq \alpha \limsup_{t \rightarrow +\infty}^* \{ \sum_{j=1}^m \Lambda_j R_j \}.$$

Using Lemma 6.6, it follows

$$(6.22) \quad \sum_{j=1}^m \Lambda_j \bar{R}_j = \limsup_{t \rightarrow +\infty}^* \{ \sum_{j=1}^m \Lambda_j R_j \} \leq \sum_{j=1}^m \Lambda_j R_j(t), \quad t \in (0, +\infty).$$

Therefore, for all $k = 1, \dots, m$, we have

$$\Lambda_k(R_k(t) - \bar{R}_k) \geq \sum_{j \neq k} \Lambda_j \bar{R}_j - \sum_{j \neq k} \Lambda_j R_j(t).$$

Moreover

$$\begin{aligned}
\Lambda_k(\underline{R}_k - \bar{R}_k) &= \liminf_{t \rightarrow +\infty}^* \{ \Lambda_k(R_k(t) - \bar{R}_k) \} \\
&\geq \liminf_{t \rightarrow +\infty}^* \{ \sum_{j \neq k} \Lambda_j \bar{R}_j - \sum_{j \neq k} \Lambda_j R_j(t) \} \geq \sum_{j \neq k} \Lambda_j \bar{R}_j - \limsup_{t \rightarrow +\infty}^* \{ \sum_{j \neq k} \Lambda_j R_j(t) \} \geq 0
\end{aligned}$$

by (6.22). Since $\Lambda_k > 0$, we conclude that $\bar{R}_k \leq \underline{R}_k$.

Step 4. We finish with the proof of the claim (6.19). For simplicity, we assume first (1.33) holds for all rows, i.e. $d_{ij} \neq 0$ for all $i, j = 1, \dots, m$, see the general case at the end of this proof. We assume without loss of generality that

$$(6.23) \quad \min_{i \neq j} -d_{ij} = 1, \quad \max_{i=1, \dots, m} d_{ii} = \frac{M}{2}.$$

4.1. We first prove the claim for R_1 . Let $t_0 > 0$ and $\phi \in C^1(0, \infty)$ such that $R_1 - \phi$ attains a maximum at t_0 and suppose that $R_1(t_0) = N_1(t_0)$. Since $N_1(t_0) \geq N_j(t_0)$, $j \geq 2$, then $\rho_{1j}(t_0) = 1$, $j \geq 2$. Thus

$$\begin{aligned}
\phi'(t_0) + \sum_{j=2}^m -d_{1j} R_1(t_0) + \sum_{j=2}^m d_{1j} N_j(t_0) &\leq 0, \text{ i.e.,} \\
\phi'(t_0) + (m-1)R_1(t_0) - \sum_{j=2}^m N_j(t_0) + \sum_{j=2}^m (1 + d_{1j})[N_j(t_0) - R_1(t_0)] &\leq 0.
\end{aligned}$$

Using (6.23) and (6.18), we have $1 + d_{1j} \leq 0$, $N_j(t_0) - R_1(t_0) \leq 0$, $j \geq 2$. The above inequality with the fact that $\sum_{j=2}^m N_j(t_0) = \sum_{j=2}^m R_j(t_0)$ lead to

$$(6.24) \quad \phi'(t_0) + (m-1)R_1(t_0) - \sum_{j=2}^m R_j(t_0) \leq 0.$$

4.2. We now prove the claim for R_k , $k \geq 2$. Let $t_0 > 0$ and $\phi \in C^1(0, \infty)$ such that $R_k - \phi$ attains a maximum at t_0 . Suppose first that $N_1(t_0) = R_k(t_0) < R_{k-1}(t_0)$, so $N_1(t_0) - \phi(t_0) = R_k(t_0) - \phi(t_0) \geq N_1(t) - \phi(t)$ for t near t_0 . It follows from the definition of $\rho_{ij}(\cdot)$ that

$$\phi'(t_0) + \sum_{j=2}^m d_{1j} \rho_{ij}(t_0) [N_j(t_0) - R_k(t_0)] \leq 0, \text{ i.e.,}$$

$$(6.25) \quad \phi'(t_0) + \sum_{j \in \mathcal{I}} d_{1j} [N_j(t_0) - R_k(t_0)] + 2 \sum_{j \in \mathcal{I}^c} d_{1j} [N_j(t_0) - R_k(t_0)] \leq 0,$$

where $\mathcal{I} := \{2 \leq j \leq m, R_k(t_0) \geq N_j(t_0)\}$ and $\mathcal{I}^c = \{2, \dots, m\} - \mathcal{I}$. Since $N_1(t_0) = R_k(t_0) < R_{k-1}(t_0)$, we obtain that $\text{card}(\mathcal{I}^c) = k-1$. We have

$$\begin{aligned} \sum_{j \in \mathcal{I}} d_{1j} [N_j(t_0) - R_k(t_0)] &= \text{card}(\mathcal{I}) R_k(t_0) - \sum_{j \in \mathcal{I}} N_j(t_0) + \sum_{j \in \mathcal{I}} (1 + d_{1j}) [N_j(t_0) - R_k(t_0)] \\ &\geq (m-k) R_k(t_0) - \sum_{j \in \mathcal{I}} N_j(t_0) = (m-k) R_k(t_0) - \sum_{j=k+1}^m R_j(t_0), \end{aligned}$$

where the last inequality follows from the fact $1 + d_{1j} \leq 0$ and $N_j(t_0) - R_1(t_0) \leq 0$, $j \in \mathcal{I}$. Noticing that $\text{card}(\mathcal{I}^c) = k-1$, we have

$$\begin{aligned} &2 \sum_{j \in \mathcal{I}^c} d_{1j} [N_j(t_0) - R_k(t_0)] \\ &= M(k-1) R_k(t_0) - M \sum_{j \in \mathcal{I}^c} N_j(t_0) + \sum_{j \in \mathcal{I}^c} (M + 2d_{1j}) [N_j(t_0) - R_k(t_0)] \\ &\geq M(k-1) R_k(t_0) - M \sum_{j \in \mathcal{I}^c} N_j(t_0) = M(k-1) R_k(t_0) - M \sum_{j=1}^{k-1} R_j(t_0), \end{aligned}$$

where the inequality follows from the fact $M + 2d_{1j} \geq 0$ and $N_j(t_0) - R_1(t_0) \geq 0$, $j \in \mathcal{I}^c$. Using these two above inequalities, it follows from (6.25) that

$$(6.26) \quad \phi'(t_0) + [M(k-1) + m-k] R_k(t_0) - M \sum_{j=1}^{k-1} R_j(t_0) - \sum_{j=k+1}^m R_j(t_0) \leq 0.$$

It remains to deal with the case $N_1(t_0) = R_k(t_0) = R_{k-1}(t_0)$. We divide it into two subcases 4.3. If $N_1(t_0) = R_k(t_0) = R_{k-1}(t_0) = \dots = R_l(t_0) < R_{l-1}(t_0)$ with $k-1 \geq l \geq 2$, then $N_1(t_0) - \phi(t_0) = R_l(t_0) - \phi(t_0) \geq N_1(t) - \phi(t)$. Applying the result in (6.26), we have

$$\phi'(t_0) + [M(l-1) + m-l] R_l(t_0) - M \sum_{j=1}^{l-1} R_j(t_0) - \sum_{j=l+1}^m R_j(t_0) \leq 0.$$

It is easy to see that

$$\begin{aligned} & [M(k-1) + m - k]R_k(t_0) - M \sum_{j=1}^{k-1} R_j(t_0) - \sum_{j=k+1}^m R_j(t_0) \\ & \leq [M(l-1) + m - l]R_l(t_0) - M \sum_{j=1}^{l-1} R_j(t_0) - \sum_{j=l+1}^m R_j(t_0). \end{aligned}$$

It follows that (6.26) holds in this case too.

4.4. If $N_1(t_0) = R_k(t_0) = R_1(t_0)$, then we have the estimate (6.24). It is easy to see that

$$\begin{aligned} & [M(k-1) + m - k]R_k(t_0) - M \sum_{j=1}^{k-1} R_j(t_0) - \sum_{j=k+1}^m R_j(t_0) \\ & \leq [m-1]R_1(t_0) - \sum_{j=2}^m R_j(t_0). \end{aligned}$$

Then (6.26) holds.

Step 5. If (1.33) only holds for a row, we will take the minimum in (6.23) among the d_{ij} 's which are nonzero and we keep zero elements of the coupling. Proceeding in a similar way as above, we obtain a new coupling satisfying (1.2) and (1.33). \square

7. PROOF OF THEOREM 1.5

We give briefly the proof of Theorem 1.5 which is based on the ideas used in the proof of Theorem 1.4. If we can find a C^1 subsolution of system (1.8), we can prove easily Theorem 1.5 as it was done in the Introduction. The existence of C^1 subsolutions of stationary equations is established in Fathi-Siconolfi [13] under quite general conditions. We think that the existence of C^1 subsolutions of system (1.8) is still true under similar conditions as in [13] but it is beyond the scope of this work. Now, we prove Theorem 1.5 using another approach.

Proof of Theorem 1.5- general case. Step 1. Approximation C^1 subsolution. Fix a Lipschitz solution V of (1.8) such that $u_i - V_i \geq 2$ where u is the solution of (1.1). The existence of V is given by Theorem 2.2. Using [13, Theorem 8.1] we can find, for all $\delta > 0$, a function $V^\delta \in C^1(\mathbb{T}^N)^m$ such that

$$\begin{aligned} (7.1) \quad & H_i(x, DV_i^\delta) + \sum_{j=1}^m d_{ij}(x)V_j^\delta \leq \delta, \quad x \in \mathbb{T}^N, \quad i = 1, \dots, m \\ & \|V_i^\delta - V_i\|_\infty \leq \delta. \end{aligned}$$

where we assumed the ergodic constant is $(0, \dots, 0)$ for simplicity. The existence V^δ is obtained by the convolution of V with a standard regularized function. It is worth noticing that the convolution of V with a regularized function is still a \mathbb{T}^N periodic function.

Step 2. Change of function. Similarly as in the proof of Theorem 1.4, we make the change of function $\exp(w_i^\delta) = u_i - V_i^\delta$. The function w^δ is solution to the new system

$$(7.2) \quad \begin{aligned} & \frac{\partial w_i^\delta}{\partial t} + F_i^\delta(x, w_i^\delta, Dw_i^\delta) + \sum_{j=1}^m d_{ij} \exp(w_j^\delta - w_i^\delta) \\ & + \exp(-w_i^\delta) [H_i(x, DV_i^\delta) + \sum_{j=1}^m d_{ij} V_j^\delta] = 0. \end{aligned}$$

with $F_i^\delta(x, w, p) = \exp(-w)H_i(x, \exp(w)p + DV_i^\delta) - \exp(-w)H_i(x, DV_i^\delta)$. We can check F_i^δ satisfies (1.28) with $\mathcal{K} = \emptyset$. Moreover, the term $\Psi(\eta)$ appearing in (1.28) can be chosen to be independent of δ . The proof relies on the upper semicontinuity of the subdifferentials of convex functions and the strict convexity of the Hamiltonians. The concrete computation is left to the reader.

Step 3. For $\eta > 0$, we define

$$(7.3) \quad \begin{aligned} M_{\eta,i}^\delta(t) &= \sup_{x \in \mathbb{T}^N, s \geq t} [w_i^\delta(x, t) - w_i^\delta(x, s) - 2\eta(s - t)], \\ M_{\eta,i}(t) &= \sup_{x \in \mathbb{T}^N, s \geq t} [w_i(x, t) - w_i(x, s) - 2\eta(s - t)], \text{ where } \exp(w_i) = u_i - V_i. \end{aligned}$$

Repeat the proof Lemma 6.4, we easily have $N_{\eta,i}^\delta := \exp(M_{\eta,i}^\delta)$ are subsolution of

$$(7.4) \quad \min\{N_{\eta,i}^{\prime\delta} + \sum_{j \neq i}^m d_{ij} \rho_{ij}^\delta(t) [N_{\eta,j}^\delta(t) - N_{\eta,i}^\delta(t)] - \delta, N_{\eta,i}^\delta - 1\} \leq 0, \quad i = 1, \dots, m,$$

where ρ_{ij}^δ is defined by

$$\rho_{ij}^\delta(t) = \begin{cases} m_1 & \text{if } N_i^\delta(t) \geq N_j^\delta(t), \\ m_2 & \text{if } N_i^\delta(t) < N_j^\delta(t), \text{ for all } i \neq j = 1, \dots, m \text{ and } t > 0, \end{cases}$$

and $m_1 \leq m_2$ are positive constants which are independent of δ .

Remark 7.1. In fact, following the proof of Lemma 6.4, we see that the only difference is the appearance of the term $A := [\exp(-w_i^\delta(\bar{x}, \bar{t})) - \exp(-w_i^\delta(\bar{y}, \bar{s}))][H_i(\bar{x}, DV_i^\delta) + \sum_{j=1}^m d_{ij} V_j^\delta(\bar{x})]$ when subtracting both sides in Step 4 as in the proof of Lemma 6.4. Thanks to the fact that $w_i^\delta(\bar{x}, \bar{t}) \geq w_i^\delta(\bar{y}, \bar{s})$ and (7.1), we have

$$-1 < \exp(-w_i^\delta(\bar{x}, \bar{t})) - \exp(-w_i^\delta(\bar{y}, \bar{s})) \leq 0 \text{ and } H_i(\bar{x}, DV_i^\delta) + \sum_{j=1}^m d_{ij} V_j^\delta(\bar{x}) \leq \delta.$$

This implies easily that $A \geq -\delta$. It explains the occurrence of the term δ in (7.4).

Since $N_{\eta,i}^\delta \rightarrow N_{\eta,i} := \exp(M_{\eta,i})$ as $\delta \rightarrow 0$, by the stability result for viscosity solutions, we can show that $N_{\eta,i}$ solves

$$(7.5) \quad \min\{N_{\eta,i}' + \sum_{j \neq i}^m d_{ij} \rho_{ij}(t) [N_{\eta,j}(t) - N_{\eta,i}(t)], N_{\eta,i} - 1\} \leq 0, \quad i = 1, \dots, m,$$

where ρ_{ij} is defined by

$$\rho_{ij}(t) = \begin{cases} m_1 & \text{if } N_i(t) \geq N_j(t), \\ m_2 & \text{if } N_i(t) < N_j(t), \text{ for all } i \neq j = 1, \dots, m \text{ and } t > 0. \end{cases}$$

Hence thanks to Lemma 6.5, we obtain that $N_{\eta,i}(t)$ converges to the same limit for all i as t tends to infinity.

Step 4. Taking a sequence $t_n \rightarrow +\infty$ such that $(u(\cdot, t_n + \cdot))_n$ converges uniformly to some function in $W^{1,\infty}(\mathbb{T}^N \times [0, \infty))^m$. Thus $(w^\delta(\cdot, t_n + \cdot))_n \rightarrow v^\delta \in W^{1,\infty}(\mathbb{T}^N \times [0, \infty))^m$ for all $\delta > 0$, and $(w(\cdot, t_n + \cdot))_n \rightarrow v \in W^{1,\infty}(\mathbb{T}^N \times [0, \infty))^m$. It is clear that $v^\delta \rightarrow v$ uniformly as $\delta \rightarrow 0$. Define

$$(7.6) \quad \begin{aligned} P_{\eta,i}^\delta(t) &= \sup_{x \in \mathbb{T}^N, s \geq t} [v_i^\delta(x, t) - v_i^\delta(x, s) - 2\eta(s - t)], \\ P_{\eta,i}(t) &= \sup_{x \in \mathbb{T}^N, s \geq t} [v_i(x, t) - v_i(x, s) - 2\eta(s - t)]. \end{aligned}$$

Since $N_{\eta,i}(t)$ converges to the same limit, we easily obtain that $P_{\eta,i}(t) = c(\eta)$ for all $t \geq 0$ and $i = 1, \dots, m$. Moreover

Lemma 7.2. *Under the assumptions of Theorem 1.5.*

(i) *The $P_{\eta,i}$'s attain their maximum at the same point for all i , see part (ii) of Lemma 6.2 for the exact definition.*

(ii) *$P_{\eta,i}(t) = 0$ for any $i = 1, \dots, m$, $t \geq 0$ and $\eta > 0$.*

We then repeat readily the proof of Theorem 1.4 to obtain the convergence as desired. \square

We end this section with the proof of Lemma 7.2.

Proof of Lemma 7.2. Proof of part (i) of Lemma 7.2.

Step 1. Fix $\tau > 0$. If $P_{\eta,j}(\tau) = 0$, then we finish the proof since we can choose $s_\tau = \tau$ and any x_τ to fulfill the requirement. We therefore assume that $P_{\eta,1}(\tau) > 0$ and let $\Phi \in C^1((0, \infty))$ such that τ is a strict maximum point of $P_{\eta,1} - \Phi$ in $[\tau - r_0, \tau + r_0]$ for some $r_0 > 0$. Since $P_{\eta,1}(\cdot)$ is a constant, $\Phi'(\tau) = 0$.

Step 2. Assume $P_{\eta,1}(\tau)$ attains its maximum at x_τ, s_τ . Consider, $x, y \in \mathbb{T}^N, t \in [\tau - r_0, \tau + r_0]$ and $s \geq t$ the test function:

$$\Psi^{1,\epsilon}(x, y, t, s) = v_1^\delta(x, t) - v_1^\delta(y, s) - 2\eta(s - t) - |x - x_\tau|^2 - |s - s_\tau|^2 - \frac{|x - y|^2}{2\epsilon^2} - \Phi(t).$$

The function $\Psi^{1,\epsilon}$ achieves its maximum over $\mathbb{T}^N \times \mathbb{T}^N \times \{(t, s)/t \leq s, t \in [\tau - r_0, \tau + r_0]\}$ at $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$. We obtain some classical estimates,

$$(7.7) \quad \begin{cases} \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} \rightarrow 0 & \text{when } \epsilon \rightarrow 0, \\ \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} (\bar{x}, \bar{s}, \bar{t}) = (x_\tau, s_\tau, \tau) & \text{since } \tau \text{ is a strict maximum point of } P_{\eta,1} - \Phi, \\ v_1^\delta(\bar{x}, \bar{t}) \geq v_1^\delta(\bar{y}, \bar{s}), \bar{s} > \bar{t} & \text{for } \epsilon, \delta \text{ are small enough since } P_{\eta,1}(\tau) > 0. \end{cases}$$

Step 3. Since v^δ is the solution of (7.2), we have

$$(7.8) \quad \begin{cases} \Phi'(\bar{t}) - 2\eta + F_1^\delta(\bar{x}, v_1^\delta(\bar{x}, \bar{t}), p) + \sum_{j=1}^m d_{1j} \exp(v_j^\delta - v_1^\delta)(\bar{x}, \bar{t}) + a(\bar{x}) \leq 0, \\ -2\eta + F_1^\delta(\bar{x}, v_1^\delta(\bar{y}, \bar{s}), p) + \sum_{j=1}^m d_{1j} \exp(v_j^\delta - v_1^\delta)(\bar{x}, \bar{s}) + O(\epsilon) + a(\bar{y}) \geq 0. \end{cases}$$

with $p = \frac{\bar{x} - \bar{y}}{\epsilon^2} + 2(\bar{x} - x_\tau)$ and

$$\begin{aligned} a(\bar{x}) &:= \exp(-v_1^\delta(\bar{x}, \bar{t})) [H_1(\bar{x}, DV_1^\delta) + \sum_{j=1}^m d_{1j} V_j^\delta(\bar{x})], \\ a(\bar{y}) &:= \exp(-v_1^\delta(\bar{y}, \bar{s})) [H_1(\bar{x}, DV_1^\delta) + \sum_{j=1}^m d_{1j} V_j^\delta(\bar{x})]. \end{aligned}$$

Step 4. Repeating Step 4 of the proof of Lemma 6.2 with taking Remark 7.1 into account. Letting ϵ tend to 0 and then δ to 0, we get

$$\sum_{j=2}^m -d_{1j} \exp(v_j - v_1)(x_\tau, \tau) \{1 - \exp[Q_{\eta,j}(\tau) - P_{\eta,1}(\tau)]\} \leq 0.$$

where $Q_{\eta,j}(\tau) := v_j(x_\tau, \tau) - v_j(x_\tau, s_\tau) - 2\eta(s_\tau - \tau)$. Since $-d_{1j} \geq 0$ for $j = 2, \dots, m$ and $Q_{\eta,j}(\tau) \leq P_{\eta,1}(\tau)$, it follows from the above inequality that $Q_{\eta,j}(\tau) = P_{\eta,1}(\tau)$, i.e., the $P_{\eta,i}$'s attain their maximum at the same point (x_τ, s_τ) .

Proof of part (ii) of Lemma 7.2.

Step 5. For any fixed $\tau > 0$, there exists (x_τ, s_τ) satisfying part (i), i.e., (x_τ, s_τ) is the common minimum point of the $P_{\eta,j}(\tau)$'s. Hence we can choose $i \in \{1, \dots, m\}$ such that

$$(7.9) \quad v_i(x_\tau, s_\tau) = \min_{j=1, \dots, m} v_j(x_\tau, s_\tau).$$

Assume that $i = 1$ and then repeat readily Steps 1, 2, 3. From (1.2), (7.9) and the fact that $\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} (\bar{x}, \bar{s}) = (x_\tau, s_\tau)$, we get

$$\sum_{j=1}^m d_{1j} \exp(v_j - v_1)(\bar{x}, \bar{s}) \leq O(\epsilon) + O(\delta).$$

This gives

$$F_1^\delta(\bar{x}, v_1^\delta(\bar{y}, \bar{s}), p) \geq \eta/2 > 0 \text{ for } \epsilon, \delta \text{ are small enough.}$$

From (7.7) and (1.28)(ii), we obtain

$$F_1^\delta(\bar{x}, v_1^\delta(\bar{x}, \bar{t}), p) - F_1^\delta(\bar{x}, v_1^\delta(\bar{y}, \bar{s}), p) \geq \Psi(\eta)(v_1^\delta(\bar{x}, \bar{t}) - v_1^\delta(\bar{y}, \bar{s})).$$

Let us recall that $\Psi(\eta)$ can be chosen to be independent of δ as mentioned in Step 2 of the proof of Theorem 1.5.

Step 6. As Step 6 in the proof of Theorem 6.1, we have

$$\sum_{j=1}^m d_{ij} \exp(v_j^\delta - v_i^\delta)(\bar{x}, \bar{t}) - \sum_{j=1}^m d_{ij} \exp(v_j^\delta - v_i^\delta)(\bar{x}, \bar{s}) \geq O(\epsilon) + O(\delta).$$

Therefore, by subtracting both sides in (7.8) and taking Remark 7.1 into account, we get

$$\Phi'(\bar{t}) + \Psi(\eta)(v_1^\delta(\bar{x}, \bar{t}) - v_1^\delta(\bar{y}, \bar{s})) + O(\epsilon) + O(\delta) \leq 0.$$

Letting ϵ tend to 0 and then δ to 0, noting that $\Phi'(\tau) = 0$, we obtain

$$\Psi(\eta)c(\eta) \leq 0, \text{ i.e. } c(\eta) \leq 0 \text{ which is a contradiction.}$$

□

8. APPENDIX

For the reader's convenience, we give a formal link between optimal control of hybrid systems with pathwise deterministic trajectories with random switching and Hamilton-Jacobi systems (1.1) with convex Hamiltonians.

Consider the controlled random evolution process (X_t, ν_t) with dynamics

$$(8.1) \quad \begin{cases} \dot{X}_t = b_{\nu_t}(X_t, a_t), & t > 0, \\ (X_0, \nu_0) = (x, i) \in \mathbb{T}^N \times \{1, \dots, m\}, \end{cases}$$

where the control law $a : [0, \infty) \rightarrow A$ is a measurable function (A is a subset of some metric space), $b_i \in L^\infty(\mathbb{T}^N \times A; \mathbb{R}^N)$, satisfies

$$(8.2) \quad |b_i(x, a) - b_i(y, a)| \leq C|x - y|, \quad x, y \in \mathbb{T}^N, \quad a \in A, \quad 1 \leq i \leq m.$$

For every a_t and matrix of probability transition $G = (\gamma_{ij})_{i,j}$ satisfying $\sum_{j \neq i} \gamma_{ij} = 1$ for $i \neq j$ and $\gamma_{ii} = -1$, there exists a solution (X_t, ν_t) , where $X_t : [0, \infty) \rightarrow \mathbb{T}^N$ is piecewise C^1 and $\nu(t)$ is a continuous-time Markov chain with state space $\{1, \dots, m\}$ and probability transitions given by

$$\mathbb{P}\{\nu_{t+\Delta t} = j \mid \nu_t = i\} = \gamma_{ij}\Delta t + o(\Delta t)$$

for $j \neq i$.

We introduce the value functions of the optimal control problems

$$(8.3) \quad u_i(x, t) = \inf_{a_t \in L^\infty([0, t], A)} \mathbb{E}_{x,i} \left\{ \int_0^t \ell_{\nu_s}(X_s, a_s) ds + u_{0, \nu_t}(X_t) \right\}, \quad i = 1, \dots, m,$$

where $\mathbb{E}_{x,i}$ denote the expectation of a trajectory starting at x in the mode i , and the functions $u_{0,i} : \mathbb{T}^N \rightarrow \mathbb{R}$, $\ell_i : \mathbb{T}^N \times A \rightarrow \mathbb{R}$ are continuous.

It is possible to show that the following dynamic programming principle holds:

$$u_i(x, t) = \inf_{a_t \in L^\infty([0, t], A)} \mathbb{E}_{x,i} \left\{ \int_0^t \ell_{\nu_s}(X_s, a_s) ds + u_{\nu_h}(X_h, t - h) \right\} \quad 0 < h \leq t.$$

Then the functions u_i satisfy the system

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sup_{a \in A} [-\langle b_i(x, a), Du_i \rangle - \ell_i(x, a)] + \sum_{j \neq i} \gamma_{ij}(u_i - u_j) = & (x, t) \in \mathbb{T}^N \times (0, +\infty), \\ u_i(x, 0) = u_{0,i}(x) & x \in \mathbb{T}^N, \end{cases} \quad i = 1, \dots, m,$$

which has the form (1.1) by setting $H_i(x, p) = \sup_{a \in A} [-\langle b_i(x, a), p \rangle - \ell_i(x, a)]$ and $d_{ii} = \sum_{j \neq i} \gamma_{ij} = 1$ and $d_{ij} = -\gamma_{ij}$ for $j \neq i$.

Remark 8.1.

- (i) Assume $\ell_i(x, a) = f_i(x)$ where the f_i 's satisfy (1.40). If the following controllability assumption is satisfied: for every i , there exists $r > 0$ such that for any $x \in \mathbb{T}^N$, the ball $B(0, r)$ is contained in $\overline{\text{co}}\{b_i(x, A)\}$. Then, Theorem 1.6 holds. Roughly speaking, it means that the optimal strategy is to drive the trajectories towards a point x^* of \mathcal{S} and then not to move anymore (except maybe a small time before t). This is suggested by the fact that all the f_i 's attain their minimum at x^* and, at such point, the running cost is smallest.
- (ii) It is also possible to consider differential games with random switchings to encompass system (1.1) with nonconvex Hamiltonians.
- (iii) More rigorous dynamical interpretations of system (1.1) are given in [23].

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